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WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. LXIX.

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IN ADDITION TO THOSE

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EDITED BY

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if, λ being any quantity,

$$\frac{a^2}{f^2}=\frac{(\delta+\lambda\delta')(\alpha+\lambda\alpha')}{(\beta+\lambda\beta')(\gamma+\lambda\gamma')}, \quad \frac{b^2}{g^2}=\frac{(\delta+\lambda\delta')(\beta+\lambda\beta')}{(\alpha+\lambda\alpha')(\gamma+\lambda\gamma')},$$

$$\frac{c^2}{h^2}=\frac{(\delta+\lambda\delta')(\gamma+\lambda\gamma')}{(\alpha+\lambda\alpha')(\beta+\lambda\beta')},$$

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$$=R\left\{\frac{1-\cos(B-C)\cos(C-A)\cos(A-B)+\cos A\cos B\cos C+\cos 2A\cos 2B\cos 2C}{2(1+\cos A\cos B\cos C)^2}\right\}^{\frac{1}{2}} \ldots 35$$

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9169. (The late W. J. C. Sharp, M.A. Suggested by Quest. 2941.) —If, in the ternarian partitionments of n, which must be of one of the forms aa^n, aab, abc, 1 be put for each of the first, 3 for each of the second, and 6 for each of the third, the sum will be $\frac{1}{2}(n-1)(n-2)$. Or generally, if the coefficient of each of the forms of the partitions of n into p parts be represented by the coefficient of that form in $(a+b+\ldots \text{ to } p \text{ terms})^p$, the sum of all the partitions will be represented by the coefficient of x^n in $x^p(1-x)^{-p}$ 71

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12556. (H. J. Woodall, A.R.C.S.)—Solve the following equations:—

(1) $x^6-6x^5+25x^4-58x^3+102x^2-96x+72=0$,

(2) $x^6-10x^5+64x^4-231x^3+616x^2-1000x+1120=0$,

(3) $x^8-10x^7+57x^6-208x^5+552x^4-1048x^3+1472x^2-1344x+768=0$,

(4) $x^{10}-40x^9+701x^8-7072x^7+45247x^6-189928x^5+520731x^4-895920x^3+886712x^2-427968x+74880=0$,

(5) $x^{12}-35x^{11}+564x^{10}-5594x^9+37990x^8-185490x^7+663479x^6-1732915x^5+3217626x^4-4065166x^3+3400100x^2-1767360x+436800=0$ 64

12680. (Professor Shields.)—Let n be any number, and let d be the difference; then

$(n^3+n^3)/(nn+d^2)=(n+n)$, $[(n^3+n)+(n^3+n)]/(nn+d^2+1)=(n+n)$,

$(n^5+n^5)/[(nn+d^2)^2+(nnd^2)]=(n+n)$ 47

12706. (Professor Shields.)—There is a large *square* piece of land AB in which is laid off another *square* CD, whose sides are oblique to the sides of the large square; and the corners of the inner square touch the sides of the large square at a distance equal to one-third of their length from the corners; the inner square, being divided by lines drawn from each of the four corners to the middle of the opposite sides, forms another square EF in the centre. These four lines and sides of the centre square are parallel to the sides of the large square. From each corner of the centre square EF, *as centres*, are drawn four equal quadrants tangent to each other, and thus enclosing $2\frac{73}{600}$ acres of land in the centre G. Find the size and area of each square, and the area of *one* quadrant 91

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13231. (Professor Finkel.)—A pole, a certain length of whose top is painted white, is standing on the side of a hill. A person at A observes that the white part of the pole subtends an angle equal to α, and on walking to B, a distance a, directly down the hill towards the foot of the pole, the white part subtends the same angle. What is the length of the white part, if the point B is at a distance b from the foot of the pole? 43

13403. (Professor Sanjána.)—If $a_z\equiv a(a+1)(a+2)\ \dots\ (a+z-1)$,

prove that $(y+x)_z = y_z+z\,.\,y_{z-1}x_1+\frac{z'z-1)}{1\,.\,2}y_{z-2}x_2+\dots+x^z$ .....(1),

$$=y_z+z(y+1)_{z-1}x_1+\frac{z(z-1)}{1.2}(y+2)_{z-2}(x-1)+\dots+(x-z+1)_z\text{.....(2)};$$

hence or otherwise show that $(q-1)/(pq-1)$

$$=\frac{(pq-q)!\,(pq-m-1)!}{(pq-1)!\,(pq-m-q)!}\left\{\frac{q-1}{pq-q-m+1}+\frac{(q-1)(q-2)(m-1)}{(pq-q-m+1)(pq-q-m+2)1!}\right.$$
$$\left.+\frac{(q-1)(q-2)(q-3)(m-1)(m-2)}{(pq-q-m+1)(pq-q-m+2)(pq-q-m+3)\,2!}+\dots\right\}\text{...(3).}$$

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[All symbols denote positive integers. Formula (3) is quoted by Rev. T. C. Simmons on p. 38 of Vol. LXII. of *Reprint*, from a solution in a previous volume. An error of transcription apparently occurs there, as the factorials in the denominator of the large bracket are omitted] ... 29

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13500. (Professor Ramaswami Aiyar, M.A.)—The natural harmonic progression being divided into the two series

$$1 + \tfrac{1}{3} + \tfrac{1}{5} + \ldots \text{ and } \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \ldots,$$

show that the sum of n terms of the former series is less than the sum of $4n$ terms of the latter series by a quantity which diminishes as n increases and vanishes when $n = \infty$ 51

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13531. (Philippa G. Fawcett.)—Prove the following rule for finding the power of the lens equivalent to n thin lenses arranged along an axis:—Take any combination out of the n lenses, and form the continued product of the powers and the distances between consecutive lenses of the combination as they stand; the sum of the products so formed for all possible combinations out of the n lenses is equal to the power of the equivalent lens. (If the combination consists of one lens only, the corresponding term is its power simply.) Extend the rule to any optical instrument symmetrical about an axis..... 44

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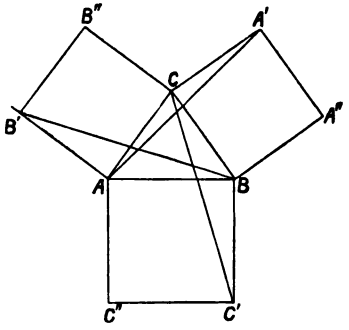

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13610. (W. J. C. Miller, B.A.)—A square is inscribed in a triangle, a second square in the triangle whose base is the side of the first square and vertex that of the given triangle, and so on *ad inf.* Find the sum of the areas of all these squares 67

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13617. (H. J. Woodall, A.R.C S.)—Mr. W. W. Rouse Ball (*Messenger of Mathematics*, XXI., pp. 34–40) quotes Euler as saying that factors of $2^p \pm 1$ are of form $np+1$. *But, if p is a given number, we can define n much more closely, and* Euler *showed that the prime factors (if any) of* $2^{31}-1$ *were necessarily primes of the form* $248n+1$ or $248n+63$. Required a proof of the latter assertion 30

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$$1-4+4^2-4^3+\ldots+4^{2r}$$

is the sum of two squares; and give their values, according as r is odd or even 47

13631. (The late Professor Sylvester, F.R.S. Jan. 31, 1897.)—Show, if ρ is any irrational p^{th} root of unity (p being a prime number), the sum of the $(p-1)$ values of $1/\{(1-\rho)(1-\rho^2)\}$ is equal to

$$\{(1-p)(p-11)\}/24 \ldots 40$$

13632. (Professor E. Lemoine.)—Soit un triangle ABC, soit K un point de son plan; AK, BK, CK coupent respectivement BC, CA, AB en α, β, γ. Par α, β, γ je mène des parallèles respectivement à CA, AB, BC qui coupent AB, BC, CA en ν, λ, μ et des parallèles à AB, BC, CA qui coupent CA, AB, BC en μ_1, ν_1, λ_1. On sait que $A\lambda$, $B\mu$, $C\nu$ concourent en un point que j'appelle ω et $A\lambda_1$, $B\mu_1$, $C\nu_1$ en un point que j'appelle ω_1. (1) Déterminer K de façon que ω et ω_1 soient sur le cercle circonscrit à ABC. (2) Trouver le lieu de K lorsque la conique A, B, C, ω, ω_1 est une hyperbole équilatère. (3) Dans ce cas quel est le lieu du centre de cette hyperbole? 39

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13634. (Professor Neuberg.)—Connaissant les centres α, β, γ, δ des cercles circonscrits aux triangles BCD, ACD, ABD, ABC formés par quatre points A, B, C, D, trouver ces points A, B, C, D 117

13636. (Professor Sanjána.)—Prove that

$$\int_0^{\frac{1}{2}\pi}\int_0^1 \tan^{-1}\frac{2m\cos\theta}{1-m^2}\,dm\,d\theta = \tfrac{1}{4}\pi^2-2\log 2,$$

where m is less than unity 39

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13644. (W. E. Jeffares, B.A.) — P is a point on the surface of the sphere from which tangent arcs are drawn to two equal non intersecting small circles, such that the product of the tangents of halves of these arcs is equal to the square of the tangent of the arc drawn from the point M midway between the centres C and C′ of the two circles. Let β be the arc MP, and θ the angle between MP and CC′. Prove that

$$\tan^2 \tfrac{1}{2}\beta = A \cos^2 \theta + B \sin^2 \theta,$$

where A and B are functions of the radius of the circles and of the distance between the centres which it is required to determine 114

13646. (Robert W. D. Christie.)—Every triangular equals the sum of three others, and therefore every integer equals the sum of $2n+1$ triangulars 48

13647. (I. Arnold.)—From a given point P outside a given triangle ABC to draw a line bisecting the triangle 42

13650. (The late Morgan Brierley. Obit Sept. 8, 1897.)—Given

$$a^2 dx^4 + y^2 dy^4 = b^2 dy^2 dx^2 + 2ay\, dy^2 dx^2,$$

to find the relation of x to y 26

13651. (R. F. Muirhead.)—In the array

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{matrix}$$

there are 40 determinants which can be formed by suppressing 2 columns and 1 row. Show that six of these can be chosen such that their vanishing involves the vanishing of all the others. Extend this to an array of p columns and q rows, and determinants of the order r, where r is less than either p or q 52

13652. (J. Woodcote West.) — Sum the series $\square_1 + \square_2 + \square_3 + \ldots$ to n terms, where $\square_n$ represents the nth square number of the form $\frac{1}{2}\{a(a+1)\}$ 51

13653. (J. C. Wasson.)—A, B, C are the vertices of a triangle inscribed in a circle. A′, B′, C′ are three other points on the circle, such that arc AA′ + arc BB′ + arc CC′ = 0 or a multiple of 2π (the algebraic sum of the arcs being taken). Then the Simson-lines of the points A′, B′, C′, with respect to the triangle ABC, meet in a point 48

13655. (W. C. Stanham.)—a and b are the semiaxes of an ellipse; $c^2 = a^2 - b^2$, $d^2 = a^2 + b^2$. From a point T on the director circle the tangents are drawn, and the corresponding normals meet at N. Show (1) that N lies on the curve

$$b^2(c^2 - dr)/a^2(c^2 + dr) = \tan^2 \theta$$

(the centre and major axis being the pole and initial line); (2) N lies within the evolute of the ellipse; (3) if the tangents to the ellipse at the feet of the other two normals from N meet at T′, T′ lies outside the rectangle formed by the tangents at the vertices of the ellipse, and between the tangents at opposite vertices 92

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13664. (Professor Rogel, LL.D.)—Prove that

$$lz < \sqrt{\{2^{\frac{1}{2}n(n-1)}(z-1)+1\}}-1 \quad (lz>0),$$

and

$$lz < \sqrt[n]{\{2^{n+1}(\tfrac{1}{2}m)^m m^a(z-1)+1\}}-1,$$

where $m+a=n$, $m=2^p$, a an integer $< 2^p$ 33

13666. (Professor Thomas Savage, M.A.)—Given in magnitude the side AB of the triangle ABC, in position and magnitude the internal bisector AD of the angle BAC, and the centre of the circumcircle of the triangle ADC, to construct the triangle ABC..... 41

13671. (R. F. Davis, M.A.)—If ABCD is a quadrilateral whose diagonals AC, BD intersect in O, and α, β, γ, δ are the circumcentres of BCD, ACD, ABD, ABC, respectively, prove that $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\alpha$ are inversely proportional to the perpendiculars from O on CD, DA, AB, BC, respectively 97

13674. (Rev. J. Cullen, B.A.)—If

$$(2p/n)^{n/(n-2)} - p\,(2p/n)^{2/(n-2)} + r = 0,$$

then

$$n^n r^{n-2} - 4p^n\,(n-2)^{n-2} = 0 \text{ ..... } 56$$

13675. (J. Brill, M.A.)—Given the eight differential equations

$$\left.\begin{aligned} du_1 &= w_1 d\phi_1 + v_1 d\psi_1 \\ dv_1 &= u_1 d\phi_1 + w_1 d\psi_1 \\ dw_1 &= v_1 d\phi_1 + u_1 d\psi_1 \end{aligned}\right\}, \qquad \left.\begin{aligned} du_2 &= w_2 d\phi_2 + v_2 d\psi_2 \\ dv_2 &= u_2 d\phi_2 + w_2 d\psi_2 \\ dw_2 &= v_2 d\phi_2 + u_2 d\psi_2 \end{aligned}\right\},$$

$$d\phi_1 + d\phi_2 = 0, \quad d\psi_1 + d\psi_2 = 0,$$

prove that

$$v_1 w_2 + v_2 w_1 + u_1 u_2 = \text{const.},$$

$$w_1 u_2 + w_2 u_1 + v_1 v_2 = \text{const.}, \quad u_1 v_2 + u_2 v_1 + w_1 w_2 = \text{const.} \text{ ..... } 64$$

13679. (H. W. Curjel, M.A.)—An elastic band passes round two fixed pegs and a pencil point. Find the curve described by the latter when the resultant pressure exerted by the band on it is constant..... 97

13681. (J. Woodcote West.)—If

$$\epsilon^{\int dy\,(P/M)} \quad \text{and} \quad \epsilon^{-\int dx\,(P/N)}$$

are both multipliers which render $M\,dx + N\,dy = 0$ integrable, then

$$\epsilon^{n\int dy\,(P/M)+(n-1)\int dx\,(P/N)}$$

is also such a multiplier 54

13682. (R. W. D. Christie.)—A father leaves a fortune of £46,189 to his four sons. The eldest gets a number of elevenths, the second thirteenths, the third seventeenths, and the fourth nineteenths, all the amounts being integral. The balance of £2,151 was given to charities. Find the amounts 54

13684. (J. C. Wasson.)—ABC is a triangle inscribed in a circle, and α, β, γ are three points on the circle such that

$$\text{arc } A\alpha + \text{arc } B\beta + \text{arc } C\gamma = (2n+1)\,\pi.$$

The nine-point circle of the triangle formed by the Simson-lines corresponding to α, β, γ is the nine-point circle of the original triangle ABC 49

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13686. (Professor J. C. Malet, F.R.S.)—Let A, B, C, D be the corners of one face of a cube, and E, F, G, H be the corners of the cube diametrically opposite to A, B, C, and D, respectively. Then, if p_1, p_2, p_3, p_4, p_5, p_6, p_7, and p_8 be perpendiculars let fall from A, B, C, D, E, F, G, and H on any plane which does not cut the cube, and a be the length of the edge of the cube, prove that

$$p_1^2+p_2^2+p_5^2+p_6^2-2p_3p_7-2p_4p_8=2a^2 \quad \ldots\ldots\ldots\ldots \quad 26$$

13687. (Lt.-Col. Allan Cunningham, R.E.)—Factorize

1,000,000,000,000,000,027 and 27,000,000,000,000,000,001 118

13688. (Professor A. Droz-Farny.)—Soit AB une corde quelconque d'une parabole P, parallèle à la tangente en un point fixe C de P. Par les points A et B on fait passer une circonférence tangente en D, à la tangente fixe du point C. Le diamètre du point D coupe AB en L et la circonférence en M. LM est égale au double du rayon de courbure de C 41

13689. (G. Hale Puckle, M.A.)—If $\phi(xy)=0$ represents a central conic, and the same assumptions are made as in Quest. 13535, No. 437, then (1) the tangents at the vertices, and (2) the parallels to them through the foci, are determined by the equations

$$\alpha(ax+hy+g)+\beta(hx+by+f)+e\Delta^{\frac{1}{2}}=0 \quad \ldots\ldots\ldots\ldots \quad (1),$$

$$\alpha(ax+hy+g)+\beta(hx+by+f)+e^2\Delta^{\frac{1}{2}}=0 \quad \ldots\ldots\ldots\ldots \quad (2)$$

..... 34

13694. (Professor Rogel, LL.D.)—Prove the convergence of the series $R \equiv \sum_{s=2,3,\ldots}^{\infty} \frac{(ls)^p}{s^r}$ $(\infty > p > 0,\ r > 1)$ 65

13695. (Professor Burnside, M.A.)—Find the differential equation of all concentric conics 55

13698. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Sum, to n terms, the series $0+4+18+48+100+180+294+$ 46

13700. (D. Biddle.)—Construct the triangle, having given (1) the radii of the three circles inscribed to the corner-spaces left by the in-circle, (2) the perpendicular distances from the mid-points of the sides to the circumcircle 56

13701. (Lt.-Col. Allan Cunningham, R.E.)—Factorize

$N_1 = 64,000,000,000,729$ and $N_2 = 729,000,000,000,064$ 117

13705. (H. W. Curjel, M.A.)—Three conics α, β, γ are drawn through D, E, F, and β, γ cut again in A; γ, α in B; and α, β in C. A fourth conic, drawn through two of the points D, E, F, cuts α in G and H, β in K and L, γ in M and N. Show that, if the conics ADEFG, BDEFK, CDEFM meet in a fourth point, then the conics ADEFH, BDEFL, CDEFN also meet in a fourth point 109

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13706. (W. E. Jeffares, B.A.)—A triangle is formed by two tangents to a parabola and their chord of contact. Prove that the line joining the intersection of the tangents to the focus is the symmedian of this triangle, and, if produced to meet the circumcircle, will be bisected at the focus 83

13707. (R. F. Davis, M.A.)—Resolve $382^4 + 1$ into factors 61

13709. (G. W. Preston, B.A.)—Construct a triangle, having given the base, the difference of the base-angles, and the sum of squares on the sides..... 63

13710. (J. Woodcote West.)—Through three points A, B, C to pass three circles, each circle to pass through two points and intersect the other circles orthogonally 82

13713. (Rev. T. Mitcheson, B.A.)—In the triangle ABC, take a point α in AB such that $A\alpha = \frac{1}{3}$ of AB, and, similarly, $B\beta = \frac{1}{3}$ of BC, and $C\gamma = \frac{1}{3}$ of CA. In the triangle $\alpha\beta\gamma$, prove, *inter alia*, (1) that each side is parallel to one median of ABC and $\frac{2}{3}$ its length, and is trisected by the other two; (2) that the centroid of both triangles is the same; and (3) that each median of $\alpha\beta\gamma$ is parallel to a side of ABC and half its length 98

13716. (W. S. Cooney.)—Squares are described on the sides of any triangle ABC (same figure as 13568). If T = area of triangle formed by intersections of AA′, BB′, CC′, T′ = area of triangle formed by the whole lines AA′, BB′, CC′, S = area of triangle formed by intersections of A″B′, B″C′, C″A′, and S′ = area of triangle formed by the whole lines A″B′, B″C′, C″A′, prove (1) that the four triangles are similar, and that $T/S = T'/S' = \frac{1}{4}$, the orthocentre of triangle formed by joining centres of squares (the point O of Quest. 13481) being a double point of T and S; (2) that the distance of any angle A of the triangle from centre of opposite square equals distance between the centres of the other two squares 94

13718. (G. Hale Puckle, M.A.)—If (with the conditions of Quest. 13572, Vol. LXVIII., p. 62) $\alpha x + \beta y = 0$, $\beta x - \alpha y = 0$ ..... (1, 2) are taken as the axes of x and y, the equation

$$ax^2 + 2hxy + by^2 + c = 0 \text{.....(3)}$$

becomes $$(a - a^2)x^2 + (b + a^2)y^2 + c = 0 \text{.....(4)};$$

and the axis of y is the axis of real foci 72

13719. (Rev. W. Allen Whitworth, M.A.)—A rod 9 feet long is to be divided into three parts, of which A. is to have the largest, B. the next, and C. the smallest. If the two fractures are made at random, A.'s, B.'s, and C.'s expectations will be respectively 66, 30, and 12 inches. But, if one fracture be made at random, and the larger portion of the rod be then divided at random, their expectations will be 64, 31, and 13 inches... 109

13721. (Professor Neuberg.) — Connaissant deux points d'une parabole et un point de la directrice, trouver le lieu géométrique du foyer 81

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13722. (Professor E. Lemoine.)—Démontrer que dans un triangle dont les trois côtés sont a, b, c, les rayons des cerçles tangents aux trois côtés r, r_a, r_b, r_c, le rayon du cercle circonscrit R, on a

$$\tfrac{1}{16}(b^2-c^2)^2(c^2-a^2)^2(a^2-b^2)^2 = [r^2(2R+r)+r_ar_br_c][r_a^2(2R-r_a)-rr_br_c][r_b^2(2R-r_b)-rr_cr_a][r_c^2(2R-r_c)-rr_ar_b]$$

..... 65

13726. (Professor K. Venkata Ramanaiya.)—Find the sum of the cubes of the first n natural numbers without assuming the sum of the squares of the first n natural numbers 78

13727. (W. J. C. Miller, B.A.)—Find *all* the values of x which will satisfy the equation

$$\cos^{-1}(3/x)+\cos^{-1}(4/x)+\cos^{-1}(5/x)=\pi \text{ ..... } 62$$

13728. (D. Biddle.)—It is well known that all square numbers are of the form $5n$ or $5n\pm1$, and that all cube numbers are of the form $7n$ or $7n\pm1$. Prove that all *prime* numbers (above 3) are of the form $6n\pm1$, and find the law of exception to the converse 86

13729. (Lt.-Col. Allan Cunningham, R.E.)—Factorize

$N_1 = 31,249,999,999,999$ and $N_2 = 31,999,999,999,999,999$ ... 73

13730. (R. Tucker, M.A.)—PQR is a circle through three co-normal points on a parabola, and $p'q'r'$ is the "N.P." circle of the tangential triangle (pqr). It is known that both these circles pass through the vertex A. Prove that the Simson-lines of A, for the triangles PQR, $p'q'r'$, pass through fixed points. Find the locus of the intersection of these lines, when (i.) P is fixed, (ii.) the rectangle under the ordinates of Q, R is constant. (*Cf.* "Some Properties of Co-normal Points on a Parabola," *Proceedings of London Mathematical Society*, Vol. xxi., pp. 442–451.) 57

13732. (Rev. W. Allen Whitworth, M.A.)—A point is taken at random on the area of (α) a circle; (β) a triangle; (γ) a square or any regular polygon; (δ) a rectangle, length $= a$, breadth $= b$; (ϵ) a parallelogram, longer side $= a$, shorter side $= b$, perpendicular breadth $= b'$. Show that the expectation of the shortest distance from the point to the boundary of the figure is (α) $\frac{1}{3}$ of radius; (β, γ) $\frac{1}{3}$ of radius of inscribed circle; (δ) $(3a-b)b \div 12a$; (ϵ) $(3a-b)b' \div 12a$ 80

13734. (The late T. S. Davies.)—If the radius of a circle be divided in extreme and mean ratio, the greater segment is the side of a regular decagon inscribed in that circle, and the sum of the squares of the radius and its greater segment is equal to the square of the side of the inscribed regular pentagon 114

13735. (R. F. Muirhead.) — (1) The roots of the equations $x^2-5x+6=0$ and $x^2-6x+5=0$ are all positive integers. Are there any other pairs of equations of the form $x^2-px+q=0$, $x^2-qx+p=0$ which have the same property? If we denote by α, β; α', β' the roots of the equation, the question may be stated in the form: Find all the positive integral solutions of the two simultaneous equations $\alpha+\beta=\alpha'\beta'$; $\alpha'+\beta'=\alpha\beta$. (2) Extend the preceding to the case of pairs of equations like $x^2\pm px\pm q=0$, $x^2\pm qx\pm p=0$ 69

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13738. (R. F. Davis, M.A.)—AB is a fixed chord of a given conic, and PQ a variable chord having a fixed direction. Prove that the foci of the parabolas through APQB lie on two fixed conics 83

13741. (W. E. Jeffares, B.A.)—Through a given point, draw a transversal across the four sides and diagonals of a quadrilateral so that it may be the centre of involution of the points of intersection 70

13743. (R. W. D. Christie.)—Prove that

$$(\omega_4+\omega_5)^{2n-1}-(\omega_2+\omega_3)^{2n-1}/(\omega_4+\omega_5)-(\omega_2+\omega_3),$$

these being the unreal roots of $x^5+1=0$, always equals two integral squares, and find them 70

13744. (J. Woodcote West.) — Express the nth number of the form which is also of the form 66

13745. (Rev. T. Mitcheson, B.A.)—In regard to the triangle ABC, whose sides are a, b, c, a point P moves so that $PB^2=PA^2+PC^2$. Determine the locus of P, and the condition of its being a real curve 66

13748. (C. E. Bickmore, M.A.)—Show that

$$\begin{vmatrix} x, & y, & z, & w \\ -w, & x, & y, & z \\ -z, & -w, & x, & y \\ -y, & -z, & -w, & x \end{vmatrix} = a^2+b^2=c^2+2d^2=e^2-2f^2,$$

a, b, c, d, e, f being definite quadratic functions of x, y, z, w 85

13756. (D. Biddle.)—Let N be a large composite odd number, whose factors are two or more primes unknown; and let $N^i=S+s$, where S is the integer and s is the decimal. Prove that, when $(S+k)^2-N=\square$, N can be split up into two factors, of which $S+k=$ half the sum; $\square=$ the square of half their difference, whilst k, an integer determined by trial, = half the difference of their differences from S. Show also to what extent the value of k depends upon that of s.

[This method enables us to split up 8616460799 by taking $k=56$ (which, owing to the nature of N in this instance, requires only 10 trials). In *Nature*, February 28, 1889, Dr. Stanley Jevons is quoted as saying that he was quite sure this number could not be factorized, as it was the product of two large primes. He was undoubtedly wrong, as indeed the writer in *Nature* (Mr. Charles J. Busk, of Blackheath) has shown, and that by using a method to which the above, though discovered independently, bears a strong resemblance] 111

13757. (G. Hale Puckle, M.A.)—If $\phi(xy)=0$ represents a parabola, show that either of the equations

$$(a+b)(ax+hy+g)+G=0, \quad (a+b)(hx+by+f)+F=0,$$

with $$(a+b)(gx+fy+c)-(A+B)/2=0,$$

will determine the focus, where $A=bc-f^2$, &c.; also, that the two first equations represent the axis, the equation to which may be written

$$a(ax+hy+g)+h(hx+by+f)=0 \text{ ..... } 74$$

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13761. (R. F. Muirhead.)—Prove that in any triangle

$$\sin A + \sin B + \sin C = 2(\cos \tfrac{1}{2}A + \cos \tfrac{1}{2}B + \cos \tfrac{1}{2}C)(\sin \tfrac{1}{2}A + \sin \tfrac{1}{2}B + \sin \tfrac{1}{2}C - 1)$$

and, more generally, for all values of A, B, C, D,

$$\sin A + \sin B + \sin C + \sin D = 2(\cos \tfrac{1}{2}A + \cos \tfrac{1}{2}B + \cos \tfrac{1}{2}C + \cos \tfrac{1}{2}D)(\sin \tfrac{1}{2}A + \sin \tfrac{1}{2}B + \sin \tfrac{1}{2}C + \sin \tfrac{1}{2}D) - 4 \sin \tfrac{1}{2}(A + B + C + D)\{\cos \tfrac{1}{2}(A + B - C - D) + \cos \tfrac{1}{2}(A + C - B - D) + \cos \tfrac{1}{2}(A + D - B - C)\}$$

..... 74

13762. (Rev. T. Mitcheson, B.A.)—Let PT be a diameter of a circle, PS a chord, and PQ, PR two other chords, one on either side of PS, and making equal angles with it. Prove that

$$\left(\frac{PQ + PR}{2PS}\right)^2 + \left(\frac{PQ - PR}{2ST}\right)^2 = 1 \quad \ldots\ldots 73$$

13763. (Rev. W. Allen Whitworth, M.A.)—There are n equal sugar-sticks. Each stick is broken into two pieces, all positions of the fracture being equally likely. Of the $2n$ pieces thus formed a child is to take the largest. Show that his expectation is $\frac{2n+1}{2(n+1)}$ of a stick... 105

13765. (Rev. J. Cullen, B.A.)—Adopting the notation $L = O_M$ to mean that L is the circumcentre of the triangle formed by the images of the point M, with respect to the sides of a given triangle ABC, prove that then $M = O_L$; also establish the following forms :—$\Omega = O_{\Omega'}$, $\Omega' = O_\Omega$; $P = O_O$, $O = O_P$; $G = O_K$, $K = O_G$; $I = O_I$, where Ω, P, &c., have their usual meanings with regard to ABC 78

13769. (J. Woodcote West.)—N is a number of the form $\alpha(\alpha+1)/2$ of the form $3\beta^2 - 3\beta + 1$; then $N = \square$ is impossible 114

13770. (E. P. Barrett, B.A.)—Let AQ be the internal bisector of the angle A in the triangle ABC, and PQ a transversal cutting AB in P and AC in R. Show that rectangle AR . BP : rect. AP . CR :: AB : AC 75

13771. (G. W. Preston, B.A.)—A, B are two given points, and CD is a given straight line. Show how to draw two lines AP, BQ to meet CD in P, Q, and to contain an angle equal to a given angle. (A, B are on same side of CD, and AP must equal BQ) 75

13773. (R. F. Davis, M.A.)—Let A, B, C, D be four points on a circle, centre O; P their mid-centre; and E, F, G the intersections of (AC, BD), (BA, CD), (AD, BC) respectively. Prove (1) the axes of the two parabolas through ABCD intersect at right angles in P, which point lies on the nine-point circle of EFG; (2) the directrices intersect in the circumcentre of EFG; (3) if Z be the mid-point of OE, and ZU, ZU' be taken along DE, such that ZU = ZU' = ZP, then UP, U'P intersect the nine-point circle of EFG again in the two foci S, S' 89

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13776. (W. C. Stanham, B.A.)—If three straight lines, one through each of the angular points of an equilateral triangle, making angles ψ_1, ψ_2, ψ_3 with the corresponding opposite sides are concurrent,

$$\cos(\psi_2+\psi_3-\psi_1)+\cos(\psi_3+\psi_1-\psi_2)+\cos(\psi_1+\psi_2-\psi_3)=0,$$

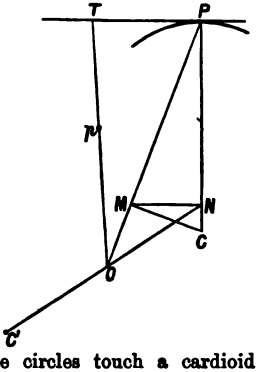
and, conversely, if this relation exists, the lines are concurrent. The following convention is adopted:—Let D, E, F be the points at which the lines meet the sides BC, CA, AB of ABC. Suppose AD, BE, CF rotated through small angles in the same one direction round A, B, C respectively, and that their new directions meet BC, CA, AB respectively at D′, E′, F′. Then $ADD'=\psi_1$, $BEE'=\psi_2$, $CFF'=\psi_3$ 106

13784. (W. J. C. Miller, B.A.)—Find the value of the definite integral $\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\theta}\cos^4(\theta+\phi)\sec^2\phi\,d\theta\,d\phi$ 105

13786. (B. H. Steede, M.A.) — Let P be any point on a given curve, and C be the corresponding centre of curvature. From C draw CM perpendicular to radius vector OP, and from foot of this perpendicular M draw MN perpendicular to the normal PC. Join N with the origin O, and produce NO, so that

$$NO : OC' = PN : k^2/p,$$

k being a constant, and p being the perpendicular from origin on tangent at P. Prove that the locus of C′ is the evolute of the reciprocal curve with respect to a circle whose centre is O and radius k 107



13789. (W. H. Salmon, B.A.)—(1) Three circles touch a cardioid and pass through its pole. Show that their other three points of intersection are collinear. (2) If two circles touch a cardioid and pass through the pole, their common chord will bisect the angle between the two lines joining the pole to the points of contact of the circles with the cardioid. (3) The locus of the other extremity of the diameter through the pole of any circle touching a cardioid and passing through the pole is a circle 119

13806. (I. Arnold.)—Bisect a given triangle by a line parallel to the base by the first book of Euclid only 108

13817. (R. Tucker, M.A.)—Ω, Ω' K are the Brocard and symmedian points of a triangle. KΩ produced meets the sides BC, CA, AB in D, E, F, and KΩ' produced meets them in D′, E′, F′. Prove that the circles A$\Omega\Omega'$, B$\Omega\Omega'$, C$\Omega\Omega'$ respectively pass through E, F′; F, D′; D, E′; also that D′F, E′D, F′E are parallel to $\Omega\Omega'$. The radical axes of A$\Omega\Omega'$ (B$\Omega\Omega'$, C$\Omega\Omega'$) and of ABC are parallel. Prove also that

$$D'F : E'D : F'E = 1/\omega_b : 1/\omega_c : 1/\omega_a, \quad \text{where} \quad \omega_a \equiv a^4-b^2c^2$$

..... 116

13833. (W. E. Jeffares, B.A.)—Show how to solve a quartic when the roots have the relation $\alpha^2\beta^2=\gamma^2\delta^2$, and find the relation between the coefficients 117

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SUPPLEMENTARY SHEET OF SOLUTIONS

IN PLACE OF

APPENDIX OF UNSOLVED QUESTIONS.

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398. (S. Watson.)—A semi-ellipse is placed in a vertical position, and a heavy particle is just set in motion at the highest point of the curve, and descends down it of its own weight; required the point at which it will leave the curve, and its velocity at that point 122

3975. (P. O'Cavanagh.)—If a double point and also a point of inflexion of a cubic be projected to infinity, what will be the nature of the new projected cubic? 125

4086. (A. Renshaw.)—If AC be the chord of a heptagon inscribed in a circle, diameter AB ($= 2r$), and AB be taken as the axis of x, and a line at right angles to it through A as that of y, then, if (x', y') be the coordinates of C, prove that

$$-\frac{2y'(2x'-r)\{x'(x'-r)-y'^2\}}{\{x'(x'-r)-y'^2\}+y'^2(2x'-r)^2}=\frac{x}{y}$$ 125

4653. (Matthew Collins, LL.B.)—Let AC be the side of a regular pentagon inscribed in a circle whose centre is O; on AO draw an equilateral triangle AOB; bisect the arc BC in D, and the arc AD in E; then prove that the chord AE will be very nearly equal to the side of a regular undecagon inscribed in the circle 128

8272. (Emily Pérrin.)—Given a triangle ABC and any point P in its plane, give a geometrical construction to determine the line through P which shall bisect the area of the triangle..... 126

12688. (D. Biddle.)—To divide a circle by parallel chords into three segments whose areas shall be in any required geometrical progression 122

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13346. (Professor P. H. Schoute.)—The ratios of the distances of a point in the plane of a triangle to the vertices A, B, C being given, find *two* points (P_1, P_2), and examine the correspondence between these points 121

13363. (R. F. Davis, M.A.)—Prove the following construction for obtaining the pair of points (P, P') satisfying the relation

PA : PB : PC $= l : m : n$ (where l, m, n are positive).

Construct the straight line $l^2a\alpha+m^2b\beta+n^2c\gamma=0$, and let fall upon it the perpendicular OX from the circumcentre. Upon OX take points P, P', such that $OP+OP'=2OX$, $OP.OP'=OA^2$ (the criterion for real roots being that any two of the quantities la, mb, nc are together greater than the third) 121

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13384. (W. J. C. Miller, B.A.)—In a given circle inscribe a quadrilateral having one side of given length, the two adjacent sides passing through a given point, and the rectangle under the distances from two given points to the intersection of its diagonals a minimum 127

13501. (W. J. C. Miller, B.A.)—Through a point, taken at random within a circle, a chord is drawn at random to cut two fixed diameters of the circle; find the probability of the triangle thus formed being acute-angled 128

13677. (W. E. Jeffares, B.A.)—If α, β, γ be the roots of

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

find the equation whose roots are

$$(\beta-\gamma)^2(2\alpha-\beta-\gamma)^2, \quad (\gamma-\alpha)^2(2\beta-\gamma-\alpha)^2, \quad (\alpha-\beta)^2(2\gamma-\alpha-\beta)^2.$$

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13714. (J. C. Wasson.)—A variable circle rolls round the circumference of a fixed circle, the contact being internal. If the linear dimensions of the variable circle are $\frac{1}{3}$ or $\frac{1}{4}$ those of the fixed circle, every diameter of the former considered rigidly attached to it has a motion identically similar to that of a Simson's line 124

13746. (Lionel E. Reay, B.A.)—Find the area of the triangle with sides equal to the medians of a given triangle. Show whether such a triangle is always possible 127

13752. (The late Professor Wolstenholme, M.A., Sc.D.)—In the two conics $U = 0$, $V = 0$, where

$$U \equiv kxy - z^2, \quad V \equiv (px-qy)(px+qy+z) + k'xy,$$

each equation has one variable parameter (k and k'); nevertheless their common chords all touch the same tricusp quartic $u^{-\frac{1}{2}} + v^{-\frac{1}{2}} + w^{-\frac{1}{2}} = 0$, where $u \equiv 2px + 2qy + \frac{1}{2}z$, $v \equiv px + qy - z + \sqrt{3}(px-qy)$, and $w \equiv px + qy - z - \sqrt{3}(px-qy)$.

The same envelope would be got by variation of k alone, or of k' alone; or we might obtain it as the envelope of the tangent at the point of contact when U, V touch one another. Explain how it is possible to have a definite envelope when any common chord must depend on the two parameters k, k' 123

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

5252. (W. J. C. MILLER, B.A.)—Find integral values of x, y, z that will satisfy each of the following equations separately:—

$$y^2-4x^2=2z,\quad y^2-4x^2=5z,\quad y^2-6x^2=10z,\quad y^2-7x^2=48z.$$

Solution by K. S. PUTNAM.

x, y, and z plainly cannot have the same integral values in all the equations. Two of them, however, may have, and the other be expressed in terms of coefficients and one of the constants.

In the general equation $y^2=ax^2+bz$, let $z=dx^2$; then

$$ax^2+bdx^2=\square,\quad a+bd=\square=V^2;\quad \text{therefore}\quad d=d(V^2-a)/b.$$

Now $a=4, 4, 6, 7$. The least value of V is therefore 3; therefore

$$z=\{(9-a)/b\}\,x^2,$$

which must be integral. Hence x^2/b must be an integer unless $9-a$ contains some factor of b, which it does in the fourth equation. Therefore x^2 must be divisible by 120, the L.C.M. of 2, 5, 10, 24.

We have, in all the equations, $x=60$ and $y=Vx=180$,

$$(1)\ z=\frac{9-a}{b}x^2=\frac{9-4}{2}x^2=9000,\qquad (2)\ z=\frac{9-a}{b}x^2=\frac{9-4}{5}x^2=3600,$$

$$(3)\ z=\frac{9-a}{b}x^2=\frac{9-6}{10}x^2=1080,\qquad (4)\ z=\frac{9-a}{b}x^2=\frac{9-7}{48}x^2=150.$$

13604. (The late Professor SYLVESTER, F.R.S. Feb. 16, 1897.)—In the triads of consecutive numbers 48, 49, 50; 98, 99, 100; 124, 125, 126, each number contains a square factor. Find how many such triads there are below the limit 1,000.

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Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Triads of successive integers of which each member contains a square factor (less than 1000):—

48, 49, 50;	475, 476, 477;
98, 99, 100;	548, 549, 550;
124, 125, 126;	603, 604, 605;
242, 243, 244, 245 (a quartett);	724, 725, 726;
342, 343, 344;	774, 775, 776;
350, 351, 352;	844, 845, 846, 847, 848 (a quintett).
423, 424, 425;	

The quartett counts as two triads, and the quintett counts as three triads, giving sixteen triads of numbers less than 1000.

13650. (The late MORGAN BRIERLEY. Obit Sept. 8, 1897.)—Given

$$a^2dx^4 + y^2dy^4 = b^2dy^2dx^2 + 2aydy^2dx^2,$$

to find the relation of x to y.

Solution by C. E. BICKMORE, M.A.; *and others.*

The equation gives $a dx^2 - y dy^2 = \pm b\,dx\,dy$;

therefore $\pm 2y\,dy = [b \pm (b^2 + 4ay)^{\frac{1}{2}}]\,dx.$

Let $b^2 + 4ay = z^2$;

therefore $y = (z^2 - b^2)/4a$ and $dy = z\,dz/2a$;

therefore $[(z^2 - b^2)\,z\,dz]/[4a^2\,(b \pm z)] = \pm dx, \quad 4a^2 dx = \pm z\,(z \pm b)\,dz$;

integrating which, $4a^2(x + \alpha) = \pm(\frac{1}{3}z^3 \pm \frac{1}{2}bz^2)$,

α being arbitrary; or

$$24a^2(x + \alpha) = \pm(4ay + b^2)\,[3b \pm (4ay + b^2)^{\frac{1}{2}}];$$

which represents four semicubical parabolas.

[The Rev. J. CULLEN solves the question as follows:—

We have $a - yp^2 = \pm bp$;

therefore $2yp = \pm b \pm (4ay + b^2)^{\frac{1}{2}}$.

Let $4ay + b^2 = u^2$.

Then $u\,(u^2 - b^2)\,du = \pm(u \pm b)\,4a^2\,dx$,

or $u\,(u \pm b)\,du = \pm 4a^2\,dx$;

therefore $2u^3 \pm 3bu^2 \pm 12a^2x + C = 0$.

The general solution is the product of these four quantities equated to zero.

13686. (Professor J. C. MALET, F.R.S.)—Let A, B, C, D be the corners of one face of a cube, and E, F, G, H be the corners of the cube

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diametrically opposite to A, B, C, and D, respectively. Then, if p_1, p_2, p_3, p_4, p_5, p_6, p_7, and p_8 be perpendiculars let fall from A, B, C, D, E, F, G, and H on any plane which does not cut the cube, and a be the length of the edge of the cube, prove that

$$p_1^2+p_2^2+p_5^2+p_6^2-2p_3p_7-2p_4p_8 = 2a^2.$$

Solution by W. S. Cooney.

If π = perpendicular from centre of cube on plane,

$$p_1+p_5 = 2\pi = p_2+p_6;$$

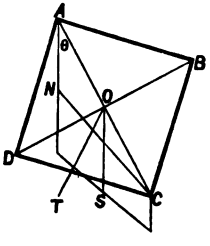
therefore $$p_1^2+p_2^2+p_5^2+p_6^2 = 8\pi^2-2p_1p_5-2p_2p_6.$$

Similarly, $$p_3^2+p_4^2+p_7^2+p_8^2 = 8\pi^2-2p_3p_7-2p_4p_8;$$

therefore
$$\begin{aligned} p_1^2+p_2^2+p_5^2+p_6^2-2p_3p_7-2p_4p_8 &= p_3^2+p_4^2+p_7^2+p_8^2-2p_1p_5-2p_2p_6 \\ &= \tfrac{1}{2}\{\Sigma p_1^2-2p_1p_5-2p_2p_6-2p_3p_7-2p_4p_8\} \\ &= \tfrac{1}{2}\{(p_1-p_5)^2+(p_2-p_6)^2+(p_3-p_7)^2+(p_4-p_8)^2\} \\ &= \tfrac{1}{2}\{\text{4 times sum of squares of perpendiculars from A, B, C, D on parallel plane through centre of cube}\} \\ &= \tfrac{1}{2}(4a^2) = 2a^2. \end{aligned}$$

This is made clear by the following consideration:—

If p, q, p', q' are perpendiculars from A, C, B, D on parallel plane through T (centre of cube), and θ, θ', ϕ are direction angles made by OS (perpendicular from O on same plane) with AC, BD, and OT, then



$$\text{AN or } p-q = \text{AC}\cos\theta = a\sqrt{2}\cos\theta,$$

$$\begin{aligned} p+q &= 2\text{OS} = 2\text{OT}\cos\phi \\ &= 2(\tfrac{1}{2}a)\cos\phi = a\cos\phi; \end{aligned}$$

therefore $$p^2+q^2 = a^2(\cos^2\theta+\tfrac{1}{2}\cos^2\phi).$$

Similarly, $$p'^2+q'^2 = a^2(\cos^2\theta'+\tfrac{1}{2}\cos^2\phi);$$

therefore $$p^2+q^2+p'^2+q'^2 = a^2(\cos^2\theta+\cos^2\theta'+\cos^2\phi) = a^2.$$

9746. (Professor Deprez.)—Dans le triangle ABC, on mène la médiane et la symédiane issues du sommet A; les droites rencontrent BC en M, N et la circonférence ABC en M', N'. Démontrer: (1) que les droites de Simson des points M', N' sont respectivement perpendiculaires aux droites AN', AM'; (2) que leurs distances à M' et N' sont égales à S^2/m^2n, S^2/m^3, S désignant l'aire ABC et m, n les longueurs AM, AN.

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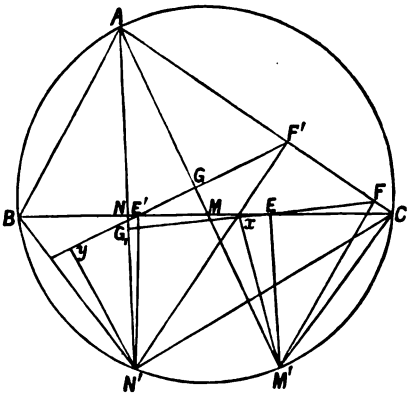
Solution by W. E. JEFFARES, B.A.

Let ABC be a triangle with its circumcircle, AMM′ and ANN′ both the median and symmedian produced to meet the circumcircle in M′ and N′; from M′ let fall M′E and M′F, perpendiculars to BC and CA; join FE and produce it to meet AN′ in G_1; again let fall N′E′ and N′F′, perpendiculars to BC and CA; join F′E′, and let it intersect AM′ in G. Now FG_1 and F′E′ are the SIMSON lines corrresponding to M′ and N′. The triangle FCM′ and G_1NE are equiangular, for



$\angle FM'C = FEC$ (since FEM′C is a cyclic quadrilateral) $= \angle G_1EN$,

and $\angle FCM' = ACB + BCM' = ACB + CAN = ENG$;

therefore $\angle NG_1E = M'CF =$ a right angle,

$M'x = M'F \sin MFE = M'F \sin M'CE$ (M′CFE being a cyclic quadrilateral) $= M'F \sin BAM$, but $M'F = AM' \sin CAM$; therefore

$$M'x = AM' \sin BAM \sin CAM.$$

Now $$\frac{\sin BAM}{\sin ABC} = \frac{a}{2m} \text{ and } \frac{\sin CAM}{\sin ACB} = \frac{a}{2m};$$

therefore $$\sin BAM \sin CAM = \frac{a^2}{4m^2} \sin B \sin C;$$

therefore $$M'x = AM' \frac{a^2 \sin B \sin C}{4m^2};$$

but $AM' \cdot AN = bc$, since the triangles AM′C and BAN′ are equiangular;

therefore $$AM = \frac{bc}{n} \text{ and } M'x = \frac{bc \cdot a^2 \sin B \sin C}{4m^2 n}$$

$$= \frac{ab \sin C \cdot ac \sin C}{4m^2 n} = \frac{4S^2}{4m^2 n} = \frac{S^2}{m^2 n}.$$

Again the triangles MGE′ and CF′N′ are equiangular, for

$$\angle ME'G = CN'F'$$

(since CF′E′N′ is a cyclic quadrilateral),

$$\angle GME' = C + MAC = C + BAN = C + BCN' = F'CN';$$

therefore $$\angle MGE' = CF'N' =$$ a right angle,

$$N'y = N'F' \sin NF'E' = N'F' \sin BCN' = N'F' \sin BAN$$

$$= AN' \sin CAN \sin BAN = (bc/m) \sin CAN \sin BAN,$$

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since $AN' \times AM = bc$ from the equiangular triangle BAN' and CAM; therefore $AN' = (bc/m)$; but, as before,

$$\sin BAN \sin CAN = \frac{a^2 \sin B \sin C}{4m^2},$$

$$N'y = \frac{a^2 bc \sin B \sin C}{4m^3} = \frac{4S^2}{4m^3} = \frac{S^2}{m^3}.$$

[This solution is shorter and simpler than that given in Vol. LXVI., p. 31.]

11935. (Professor LEMOINE.)—La somme des carrés de trois nombres entiers, multipliée par la somme des doubles produits des carrés de ces nombres, est une somme de trois carrés entiers.

Solution by R. TUCKER, M.A.

We have to prove, if l, m, n are *any* numbers, that

$$(l^2+m^2+n^2)(m^2n^2+n^2l^2+l^2m^2) \equiv \text{sum of 3 squares}.$$

Now sinister side

$$= l^2\left[m^2+mn+n^2\right]^2 + m^2\left[l^2-mn\right]^2 + n^2\left[l^2-mn\right]^2; \quad \therefore \text{ \&c.}$$

In the cases

$l=1,\ m=2,\ n=3 \mid u=686,$	$= 1^2.19^2 + 2^2.5^2 + 3^2.5^2 = 361+100+225,$
$l=2,\ m=3,\ n=1$	$= 2^2.13^2 + 3^2.1^2 + 1^2.1^2 = 676+9+1,$
$l=3,\ m=1,\ n=2$	$= 3^2.7^2 + 1^2.7^2 + 2^2.7^2 = 441+49+196.$

13403. (Professor SANJÂNA.) — If $a_z \equiv a(a+1)(a+2)\ldots(a+z-1)$,

prove that $$(y+x)_z = y_z + z.y_{z-1}x_1 + \frac{z(z-1)}{1.2}y_{z-2}x_2 + \ldots + x^z \quad \ldots\ldots\ldots(1),$$

$$= y_z + z(y+1)_{z-1}x_1 + \frac{z(z-1)}{1.2}(y+2)_{z-2}(x-1) + \ldots + (x-z+1)_z \quad \ldots(2);$$

hence or otherwise show that $(q-1)/(pq-1)$

$$= \frac{(pq-q)!\,(pq-m-1)!}{(pq-1)!\,(pq-m-q)!}\left\{\frac{q-1}{pq-q-m+1} + \frac{(q-1)(q-2)(m-1)}{(pq-q-m+1)(pq-q-m+2)1!}\right.$$

$$\left. + \frac{(q-1)(q-2)(q-3)(m-1)(m-2)}{(pq-q-m+1)(pq-q-m+2)(pq-q-m+3)\,2!} + \ldots\right\} \ldots(3).$$

[All symbols denote positive integers. Formula (3) is quoted by Rev. T. C. SIMMONS on p. 38 of Vol. LXII. of *Reprint*, from a solution in a previous volume. An error of transcription apparently occurs there, as the factorials in the denominator of the large bracket are omitted.]

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Solution by Professor S. SIRCOM, M.A.

By the same method as is used to prove LEIBNITZ's theorem, we have

$$\left(u^2\frac{d}{du}\right)^z vw = v\left(u^2\frac{d}{du}\right)^z w + z\left(u^2\frac{d}{du}\right)^{z-1} w\left(u^2\frac{d}{du}\right) v + \frac{z(z-1)}{1.2}\left(u^2\frac{d}{du}\right)^{z-2} w\left(u^2\frac{d}{du}\right)^2 v + \ldots + w\left(u^2\frac{d}{du}\right)^z v.$$

Let $v = u^x$, $w = u^y$; then, putting $u = 1$ after the differentiations, $\left(u^2\frac{d}{du}\right)^z vw$ becomes $(y+x)_z$; so for the rest; whence (1) is proved. It is also proved inductively by multiplying by $y+x+z$.

Similarly,

$$\left(\frac{d}{du}\right)^z vw = v\left(\frac{d}{du}\right)^z w + z\left(\frac{d}{du}\right)^{z-1} w\left(\frac{d}{du}\right) v + \frac{z(z-1)}{1.2}\left(\frac{d}{du}\right)^{z-2} w\left(\frac{d}{du}\right)^2 v + \ldots + w\left(\frac{d}{du}\right)^z v.$$

Let $v = u^x$, $w = u^{y+z-1}$; then, putting $u = 1$ after the differentiations, $\left(\frac{d}{du}\right)^z u^{y+z+z-1}$ becomes

$$(y+x+z-1)(y+x+z-2)\ldots(y+x) = (y+x)_z,$$

and $\left(\frac{d}{du}\right)^{z-1} u^{y+z-1}$ becomes $(y+z-1)(y+z-2)\ldots(y+1) = (y+1)_{z-1}$,

and so for the rest, which gives (2).

In (2) let $y = pq-q-m+2$, $x = q-2$, $z = m-1$; then

$$(y+x)_z = (pq-m)_{m-1} = (pq-q-m+2)_{m-1} + \frac{m-1}{1}(pq-q-m+3)_{m-2}(q-2) + \frac{(m-1)(m-2)}{1.2}(pq-q-m+4)_{m-3}(q-2)(q-3) + \ldots,$$

which gives (3) on multiplying by $q-1$ and introducing the factorials.

13617. (H. J. WOODALL, A.R.C.S.)—Mr. W. W. ROUSE BALL (*Messenger of Mathematics*, XXI., pp. 34–40) quotes EULER as saying that factors of $2^p \pm 1$ are of form $np+1$. *But, if p is a given number, we can define n much more closely, and* EULER *showed that the prime factors (if any) of $2^{31}-1$ were necessarily primes of the form $248n+1$ or $248n+63$.* Required a proof of the latter assertion.

Solution by H. W. CURJEL, M.A.

Since 2 is a quadratic residue of any divisor of 2^p-1, the divisors are of the form $8n\pm1$ (MATHEWS, p. 38, or DIRICHLET, p. 91). Therefore

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all the divisors are of the form $8np + 4p \pm 2p + 1$ (according as $p \equiv 1$ or 3, mod 4), or $8np + 1$; *i.e.*, if $p = 31$, of the form $248n + 1$ or $248n + 63$.

[Lt.-Col. ALLAN CUNNINGHAM solves the Question, in not quite similar terms, as follows:—To show that the *prime* divisors of $(2^{31} - 1)$ are of form $(248n + 1)$ or $(248n + 63)$.

Let $N = (2^p - 1)$, where p is *prime*. Therefore $N = 2 \cdot 2^{p-1} - 1$, wherein 2^{p-1} is a square, since $(p - 1)$ is an *even* number. Therefore 2 is a 2-ic residue of every prime factor (F) of N. Therefore every F is of one of the forms $(8f + 1)$, $(8f + 7)$. But EULER showed that every F was of form $(mp + 1)$. Combining, then, every F is of form $(8mp + 1)$, or else of both forms $(8p + 7)$, $(mp + 1)$. When $p = 31$, this gives, every F is of form $(8 \cdot 31n + 1)$, or of form $(8 \cdot 31n + 63)$.]

13411. (Professor BEYENS.)—A person, standing on a level plain, observes the angles of elevation of two distant towers. He then walks towards them, always in a direction midway between the directions of the two towers. When exactly in the line between the two towers, he observes their elevations to be l and m times their former values. Prove that, at the latter observation, the distances of the two towers are in the ratio of $m - 1 : l - 1$ nearly.

Solution by Rev. J. CULLEN, B.A.

Let r and r_1 be the distances of the towers. Then, since the tangent to his path bisects the angle between r and r_1, it is clear he describes the hyperbola

$$r - r_1 = 2a,$$

$$r = h \cot \alpha, \quad h = x \tan m\alpha,$$

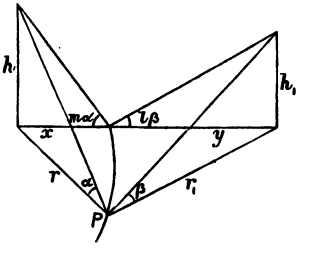
and, as

$\tan m\alpha > m\alpha > \sin m\alpha$, $r = mx$ (nearly) if the angles are small. Similarly,

$$r_1 = ly; \quad \therefore \quad mx - ly = 2a.$$

Also $x - y = 2a$;

$$\therefore \quad (m-1)x = (l-1)y, \quad \text{or} \quad y : x :: m-1 : l-1.$$



13620. (R. KNOWLES, B.A.)—A triangle ABC is inscribed in a conic; prove that (1) the three lines joining the poles of BC, AC, AB to A, B, C, respectively, meet in the same point O; (2) the three points of intersection of BC, AC, AB with the tangents at A, B, C, respectively, are on the polar of O.

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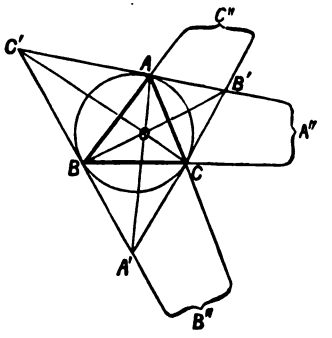
Solution by W. J. Dobbs, M.A.

Project the conic into a circle. Then B′C′, C′A′, A′B′ are *isogonal* to the straight lines drawn through A, B, C, respectively, parallel to the opposite sides. Therefore A″, B″, C″ are collinear. Therefore triangles ABC, A′B′C′ are *in perspective*, O being the *centre of perspective*. The polars of A and A′ both pass through A″. Therefore A″, and similarly B″ and C″, lie on the polar of O.



[Mr. Curjel solves the Question as follows:—

Let the tangents at A, B, C form the triangle PQR, and let p, q, r be the lengths of the diameters parallel to QR, RP, PQ. Then

$$\frac{PB \cdot RA \cdot QC}{PC \cdot RB \cdot QA} = -\frac{q \cdot p \cdot r}{r \cdot q \cdot p} = -1.$$

Therefore PA, QB, RC meet in a point O. Again the point D, where BC meets RQ, is the pole of PA (for BC is divided harmonically by D and PA). Therefore D lies on the polar of O; similarly, the other two points lie on the polar of O.

Professor A. Droz-Farny makes the following *remarque*:—"Dans le cas d'une circonférence circonscrite, le point O et la droite $\alpha\beta\gamma$ ($\equiv$ A″B″C″) deviennent respectivement le point et la droite de Lemoine.

The Proposer adds the following:—"This is a modification of Pascal's theorem. A triangle is the simplest inscribed figure. It is a fact that the points of intersection of the opposite sides of *any* polygon inscribed in a conic lie on the polar of the point of intersection of the diagonals of the circum-polygon, *except when more than one pair of its opposite sides are parallel, and the centre of the conic has no polar*. Some contributors have disputed the truth of these statements, but I believe some time they will be recognised truths.

13586. (R. W. Chapman, M.A., B.C.E.)—Any three circles are drawn with A, B, and C as centres. Show that the common radical axis of the three circles of similitude of these circles taken in pairs passes through the centre of the circle circumscribing the triangle ABC.

Solution by the Proposer; *and* W. E. Jeffares, B.A.

Let the equations to the three circles be

$$x^2 + y^2 - r^2 = 0, \quad (x-a_1)^2 + (y-b_1)^2 - r_1^2 = 0 \quad \ldots\ldots\ldots\ldots(1, 2),$$

$$(x-a_2)^2 + (y-b_2)^2 - r_2^2 = 0 \quad \ldots\ldots\ldots\ldots(3).$$

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The circle of similitude of (1) and (2) is

$$r_1^2(x^2+y^2-r^2)-r^2\{(x-a_1)^2+(y-b_1)^2-r_1^2\}=0.$$

The circle of similitude of (1) and (3) is

$$r_2^2(x^2+y^2-r^2)-r^2\{(x-a_2)^2+(y-b_2)^2-r_2^2\}=0.$$

Writing down the equation to the radical axis of these two circles, and simplifying, by ordinary processes it reduces to the form

$$\{2a_1x+2b_1y-(a_1^2+b_1^2)\}(r^2-r_2^2)+\{2a_2x+2b_2y-(a_2^2+b_2^2)\}(r_1^2-r^2)=0.$$

This clearly passes through the point of intersection of the lines

$$2a_1x+2b_1y-(a_1^2+b_1^2)=0 \quad \text{and} \quad 2a_2x+2b_2y-(a_2^2+b_2^2)=0.$$

These are the lines through the middle points of AB and AC perpendicular to the sides. Therefore the radical axis passes through the circumcentre of ABC.

[Mr. CULLEN solves the Question as follows:—Since A and B are harmonic conjugates of the centres of similitude, it follows that any circle through A and B intersects the circle of similitude orthogonally. Hence the circumcircle of the triangle ABC intersects the three circles of similitude orthogonally. Therefore the circumcentre lies on the common radical axis, since tangents from it are equal.]

13664. (Professor ROGEL, LL.D.)—Prove that

$$lz < \sqrt{\{2^{\frac{1}{2}n(n-1)}(z-1)+1\}}-1 \quad (lz>0),$$

and

$$lz < \sqrt[n]{\{2^{a+1}(\tfrac{1}{2}m)^m m^a(z-1)+1\}}-1,$$

where $m+a=n$, $m=2^p$, a an integer $< 2^p$.

Solution by the PROPOSER.

If $lz>0$, we have

$$z > 1+lz+(1/2!)(lz)^2\ldots+(1/n!)(lz)^n;$$

$$\therefore n!z > 1+\binom{n}{1}lz+\binom{n}{2}(lz)\ldots+\binom{n}{n-1}(lz)^{n-1}+\binom{n}{n}(lz)^n+n!-1,$$

or

$$n!z < (1+lz)^n+n!-1;$$

therefore

$$lz < \sqrt[n]{\{n!(z-1)+1\}}-1 \quad \ldots\ldots\ldots\ldots (1).$$

In connexion with $n! < 2^{\frac{1}{2}n(n-1)}$ (E. LAMPE and W. J. C. MILLER, *Educational Times*, Vol. LIV., page 51), we find

$$lz < \sqrt[n]{\{2^{\frac{1}{2}n(n-1)}(z-1)+1\}}-1 \quad \ldots\ldots\ldots\ldots (2),$$

and putting $lz < (2z-1)^{\frac{1}{2}}-1$, $z>1$. According to the inequation

$$(m+a)! < 2^{a+1}(\tfrac{1}{2}m)^m m^a, \quad m+a=n, \quad m=2^p, \quad a \text{ integer} < 2^p,$$

found by E. LAMPE, it consists

$$lz < \sqrt[n]{\{2^{a+1}(\tfrac{1}{2}m)^m m^a(z-1)+1\}}-1 \quad \ldots\ldots\ldots\ldots (3),$$

and setting $a=0$, $lz < \frac{1}{2}m\sqrt[m]{(2z)}-1$, $z>1$.

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13689. (G. Hale Puckle, M.A.)—If $\phi(xy) = 0$ represents a central conic, and the same assumptions are made as in Quest. 13535, No. 437, then (1) the tangents at the vertices, and (2) the parallels to them through the foci, are determined by the equations

$$\alpha(ax+hy+g)+\beta(hx+by+f)+e\Delta^{\frac{1}{2}} = 0 \quad \ldots\ldots\ldots\ldots (1),$$

$$\alpha(ax+hy+g)+\beta(hx+by+f)+e^2\Delta^{\frac{1}{2}} = 0 \quad \ldots\ldots\ldots\ldots (2).$$

Solution by the Proposer.

Let the equation to a line in case (2) be

$$\alpha(ax+hy+g)+\beta(hx+by+f)+k = 0;$$

then, since it passes through the focus,

$$-\alpha^2\Delta^{\frac{1}{2}}/\rho^2-\beta^2\Delta^{\frac{1}{2}}/\rho^2+k = 0, \qquad k = (\alpha^2+\beta^2)\Delta^{\frac{1}{2}}/\rho^2 = e^2\Delta^{\frac{1}{2}}.$$

The tangent at the vertex is the locus of a point, the perpendicular from which on (2) is equal to the perpendicular on the directrix multiplied by e; and the equation to this locus is (1).

6388. (W. R. Roberts, M.A.)—Prove that the line whose coordinates are a, b, c, d, e, f, g, h is a "line through two points" of the quartic curve $\alpha x^2+\beta y^2+\gamma z^2+\delta w^2 = 0$, $\alpha' x^2+\beta' y^2+\gamma' z^2+\delta' w^2 = 0$,

if, λ being any quantity,

$$\frac{a^2}{f^2} = \frac{(\delta+\lambda\delta')(\alpha+\lambda\alpha')}{(\beta+\lambda\beta')(\gamma+\lambda\gamma')}, \quad \frac{b^2}{g^2} = \frac{(\delta+\lambda\delta')(\beta+\lambda\beta')}{(\alpha+\lambda\alpha')(\gamma+\lambda\gamma')},$$

$$\frac{c^2}{h^2} = \frac{(\delta+\lambda\delta')(\gamma+\lambda\gamma')}{(\alpha+\lambda\alpha')(\beta+\lambda\beta')},$$

$$(\alpha f^2+\beta g^2+\gamma h^2)+\lambda(\alpha' f^2+\beta' g^2+\gamma' h^2) = 0.$$

Solution by H. J. Woodall, A.R.C.S.

This may be obtained thus (Salmon, *Three Dimensions*, 4th ed., p. 193):—When the line whose ray coordinates are a, b, c, f, g, h is a generator of

$$(\alpha+\lambda\alpha')x^2+(\beta+\lambda\beta')y^2+(\gamma+\lambda\gamma')z^2+(\delta+\lambda\delta')w^2 = 0,$$

then we have

$$0 = \ldots(\gamma+\lambda\gamma')b^2+(\beta+\lambda\beta')c^2+(\delta+\lambda\delta')f^2 \ldots\ldots\ldots\ldots (1),$$

$$0 = (\gamma+\lambda\gamma')a^2+\ldots+(\alpha+\lambda\alpha')c^2+\ldots+(\delta+\lambda\delta')g^2 \ldots\ldots\ldots\ldots (2),$$

$$0 = (\beta+\lambda\beta')a^2+(\alpha+\lambda\alpha')b^2+\ldots+(\delta+\lambda\delta')h^2 \ldots\ldots\ldots\ldots (3),$$

$$0 = \ldots(\alpha+\lambda\alpha')f^2+(\beta+\lambda\beta')g^2+(\gamma+\lambda\gamma')h^2 \ldots\ldots\ldots\ldots (4),$$

which are equivalent to the four equations given in the Question. Now a generator of the surface $(\alpha+\lambda\alpha')x^2+$ &c. $= 0$ is a "line through two points" of the common intersection of the two given quadrics; whence follows the theorem.

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6566. (The late Professor WOLSTENHOLME, Sc.D.)—The tangents at A, B, C to the circle ABC form a triangle A′B′C′; AA′, BB′, CC′ concur in O′, and O is the centre of the circle ABC; prove that OO′

$$= R\left\{\frac{1-\cos(B-C)\cos(C-A)\cos(A-B)+\cos A\cos B\cos C+\cos 2A\cos 2B\cos 2C}{2(1+\cos A\cos B\cos C)^2}\right\}^{\frac{1}{2}}.$$

Solution by Professor SANJÁNA.

The distance required is that between the circumcentre and the symmedian point of the triangle ABC, and is known to be $R(1-3\tan^2\omega)^{\frac{1}{2}}$. See MILNE, *Companion*, p. 122.

$$\text{Now } 1-3\tan^2\omega = \frac{\operatorname{cosec}^2\omega-4}{\operatorname{cosec}^2\omega-1} = \frac{\operatorname{cosec}^2 A+\operatorname{cosec}^2 B+\operatorname{cosec}^2 C-4}{\operatorname{cosec}^2 A+\operatorname{cosec}^2 B+\operatorname{cosec}^2 C-1}$$
$$= \frac{\Sigma\sin^2 B\sin^2 C-4\Pi\sin^2 A}{\Sigma\sin^2 B\sin^2 C-\Pi\sin^2 A} \equiv \frac{N}{D}.$$

$$\text{And } D = (1-\sin^2 A)(1-\sin^2 B)(1-\sin^2 C)+\sin^2 A+\sin^2 B+\sin^2 C-1$$
$$= \cos^2 A\cos^2 B\cos^2 C+1+2\cos A\cos B\cos C, \text{ as } A+B+C=\pi,$$
$$= (1+\cos A\cos B\cos C)^2;$$

also

$$N = \tfrac{1}{4}\{\Sigma(1-\cos 2B)(1-\cos 2C)-2(1-\cos 2A)(1-\cos 2B)(1-\cos 2C)\}$$
$$= \tfrac{1}{4}\{1-\cos 2B\cos 2C-\cos 2C\cos 2A-\cos 2A\cos 2B+2\cos 2A\cos 2B\cos 2C)\}$$
$$= \tfrac{1}{4}\{1-\tfrac{1}{2}\Sigma\cos(2B+2C)-\tfrac{1}{2}\Sigma\cos(2B-2C)+2\cos 2A\cos 2B\cos 2C\}$$
$$= \tfrac{1}{4}\{1-\tfrac{1}{2}(-4\cos A\cos B\cos C-1)-\tfrac{1}{2}[4\cos(B-C)\cos(C-A)\cos(A-B)-1]+2\Pi\cos 2A\}$$
$$= \tfrac{1}{4}\{2+2\cos A\cos B\cos C-2\cos(B-C)\cos(C-A)\cos(A-B)+2\cos 2A\cos 2B\cos 2C\}.$$

The result follows at once from these values of N and D. For another solution, see *Reprint*, Vol. LXII., p. 109.

7882. (W. J. C. SHARP, M.A.)—Prove that the equation

$$\frac{d^2y}{dx^2}+\phi(x)\frac{dy}{dx}+\psi(x)\,y = 0$$

is reducible to an equation with constant coefficients, if

$$\frac{d}{dx}\left\{\frac{\psi'(x)+2\phi(x)\psi(x)}{[\psi(x)]^{\frac{3}{2}}}\right\} = 0.$$

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Solution by Rev. J. CULLEN, B.A.

Take z for the independent variable, and we have

$$\frac{d^2y}{dz^2} + \frac{z'' + z'}{z'^2}\phi(x)\frac{dy}{dz} + \frac{\psi(x)}{z'^2}y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where dashes represent differentiation with respect to x.

(1) Will have constant coefficients if

$$\frac{z'' + z'}{z'^2}\phi(x) = a, \quad \frac{\psi(x)}{z'^2} = b \quad \ldots\ldots\ldots\ldots\ldots\ldots(2, 3).$$

From (2) we get $\qquad az' = z''/z' + \phi(x),$

which gives on integration

$$az = \log z' + \int \phi(x)\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Eliminating z' between (3) and (4), the latter becomes

$$az = \tfrac{1}{2}\log\psi(x) - \tfrac{1}{2}\log b + \int \phi(x)\,dx \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Differentiating (5) and employing (3), we find

$$\frac{a}{b^{\frac{1}{2}}}\{\psi(x)\}^{\frac{1}{2}} = \tfrac{1}{2}\frac{\psi'(x)}{\psi(x)} + \phi(x).$$

That is,

$$\frac{2a}{b^{\frac{1}{2}}} = \frac{\psi'(x) + 2\psi(x)\phi(x)}{\{\psi(x)\}^{\frac{1}{2}}}. \quad \therefore \text{ \&c.}$$

3798. (A. MARTIN.)—Supposing the earth to be an oblate spheroid whose semi-axes are a and b, find the mean distance of its centre from all points in its surface.

Solution by H. J. WOODALL, A.R.C.S.

Surface of the spheroid $= 2\int_0^a 2\pi y\{1 + (dy/dx)^2\}^{\frac{1}{2}}\,dx,$

where $\qquad y = b(a^2 - x^2)^{\frac{1}{2}}/a, \quad dy/dx = -bx/\{(a^2 - x^2)^{\frac{1}{2}}a\}.$

Substituting, we get $= 4\pi b\int_0^a \{a^4 - x^2(a^2 - b^2)\}^{\frac{1}{2}}\,dx/a^2$

$$= 2\pi a^2 b \text{ arc sin } \{(a^2 - b^2)^{\frac{1}{2}}/a\}/(a^2 - b^2)^{\frac{1}{2}} + 2\pi b^2.$$

[If $b = a$, this becomes $4\pi a^2$.]

The required mean distance

$$= 2\int_0^a (x^2 + y^2)^{\frac{1}{2}} 2\pi y\{1 + (dy/dx)^2\}^{\frac{1}{2}}\,dx/\text{surface of spheroid.}$$

The numerator reduces to

$$4\pi b/a^3 \int_0^a \{a^2b^2 + x^2(a^2 - b^2)\}^{\frac{1}{2}}\{a^4 - x^2(a^2 - b^2)\}\,dx,$$

which can be reduced to elliptic integrals.

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13403. (Professor NANSON.)—If u, u', are essentially positive quadratic functions of x, $\int_{\infty}^{+\infty} \log \frac{u'}{u} \frac{dx}{u} = \frac{\pi}{\sqrt{\Delta}} \log \frac{H + 2\sqrt{(\Delta\Delta')}}{4\Delta}$,

where Δ, Δ' are the discriminants of u, u' and H is the invariant intermediate to Δ, Δ'.

Solution by Rev. J. CULLEN, B.A.

Let $u = ax^2 + 2bx + c$, $u' = a'x^2 + 2b'x + c'$.

Then $\Delta = ac - b^2$, $\Delta' = a'c' - b'^2$, $H = ac' + a'c - 2bb'$.

Putting $ax + b = \sqrt{\Delta} \tan\theta$, we have

$$au = \Delta \sec^2\theta, \quad adx = \sqrt{\Delta} \sec^2\theta \, d\theta, \quad \text{or} \quad dx/u = d\theta/\sqrt{\Delta};$$

$$\therefore \quad I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \left[\log(\alpha\cos 2\theta + \beta \sin 2\theta + \tfrac{1}{2}H) - \log\Delta\right] d\theta/\sqrt{\Delta},$$

where $\alpha = \frac{1}{2}H - a'\Delta/a$ and $\beta = \sqrt{\Delta}\,(ab' - a'b)/a$.

$$\therefore \quad I\sqrt{\Delta} + \pi \log\Delta = \tfrac{1}{2}\int_{-\pi}^{\pi} \log(\alpha\cos\phi + \beta\sin\phi + \tfrac{1}{2}H)\, d\phi.$$

Hence, writing $\alpha = r\cos\gamma$, $\beta = r\sin\gamma$, and $\phi - \gamma = \psi$, we get

$$I\sqrt{\Delta} + \pi\log\Delta = \tfrac{1}{2}\int_{-\pi-\gamma}^{\pi-\gamma} \log(r\cos\psi + \tfrac{1}{2}H)\, d\psi$$

$$= \pi \log\left\{\frac{\frac{1}{2}H + \sqrt{(\frac{1}{4}H^2 - r^2)}}{2}\right\},$$

since, in the expansion, terms containing cosines of the multiples of ψ vanish. Whence, substituting for r, we have

$$I = \frac{\pi}{\sqrt{\Delta}} \log \frac{H + 2\sqrt{(\Delta\Delta')}}{4\Delta}.$$

13561. (P. W. FLOOD.)—Solve

$$2bx - ax = x^4/a^2 + (2x^2/a)\,(b - x).$$

Solution by L. E. REAY, B.A.; F. H. PEACHELL, B.A.; *and others.*

$$x^4/a^2 + (2x^2/a)(b-x) = 2bx - ax, \quad (x^2/a - x + b)^2 = x^2 - ax + b^2,$$

$$(x^2/a - x)^2 + 2b\,(x^2/a - x) = a\,(x^2/a - x);$$

therefore $x^2/a - x = 0$ or $x^2/a - x + 2b - a = 0$;

therefore roots of equation are 0, a, and the roots of the quadratic

$$x^2 - ax + 2ab - a^2 = 0.$$

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12417. (Professor HAYWARD.)—Find two numbers such that each of them and also their sum and their difference when increased by unity shall be square numbers.

Solution by R. W. D. CHRISTIE.

We have $x+1=a^2$, $y+1=b^2$, $x+y+1=c^2$, $x-y+1=d^2$;

therefore $x+y+2=a^2+b^2=c^2+1$ and $x-y=a^2-b^2=d^2-1$;

therefore $2x+2=2a^2=c^2+d^2$;

therefore $x+1=a^2=\frac{1}{2}(c^2+d^2)=\frac{1}{2}\left[(e+f)^2+(e-f)^2\right]=e^2+f^2$;

similarly $y+1=b^2=2ef+1$.

Let $e=g^2-h^2$ and $f=2gh$.

Then $x+1=(g^2+h^2)^2=$ a square .....(I.),

and $y+1=4gh\,(g^2-h^2)+1=(1+2gh)^2$, say ..... (II.).

Working out, we get $g=\frac{1}{2}h\pm\frac{1}{2}\left[(5h^2+4)^{\frac{1}{2}}\right]$.

Solving $(5h^2+4)^{\frac{1}{2}}=k$ by convergents, we get values for (I.) and (II.), viz.,

$h=1, 1, 2, 3, 5, 8, 13, 21, 34,$ &c.,

$g=1, 2, 3, 5, 8, 13, 21, 34, 55,$ &c.,

where the law of continuation is $a_n+a_{n-1}=a_{n+1}$, and finally

$x=3, 24, 168, 1155, 7920, 54288, 372099,$ &c.,

$y=0, 24, 120, 960, 6240, 43680, 297024,$ &c.

Let p, q be any two *consecutive* values of the "continuant" series h or g, given above; more general values are

$$x=\left[(p^2+q^2)^2+\left\{q\,(q-2p)\right\}^2\right]^2-1,$$
$$y=\left\{1\pm 2q\,(p^2+q^2)(q-2p)\right\}^2-1.$$

13633. (Professor CROFTON, F.R.S.)—If z is a function of x, y which satisfies the condition $x\,(dz/dx)-y\,(dz/dy)=az$, show that $d^2z/(dx\,dy)$ also satisfies it.

Solution by H. W. CURJEL, M.A.; LIONEL E. REAY, B.A.; *and* Rev. J. CULLEN, B.A.

Differentiating the equation with respect to x, we have

$$(dz/dx)+x\,(d^2z/dx^2)-y\left[d^2z/(dx\,dy)\right]=a\,(dz/dx);$$

differentiating this result with respect to y,

$$x\,(d/dx)\left[d^2z/(dx\,dy)\right]-y\,(d/dy)\left[d^2z/(dx\,dy)\right]=a\left[d^2x/(dx\,dy)\right].$$

Hence the required result.

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2044. (Professor WOLSTENHOLME, D.Sc.)—The mean value of the distance from one of the foci of all points within a given prolate spheroid is $a\frac{3+e^2}{4}$.

Solution by G. B. M. ZERR.

Let $r = \frac{a(1-e^2)}{1-e\cos\theta}$ be the equation to the ellipse. Let M = the required mean. Then

$$\begin{aligned} M &= \iint r\,dv \Big/ \iint dv = 2\pi \int_\pi^0 \int_0^r r^3 \sin\theta\,d\theta\,dr \Big/ \tfrac{4}{3}\pi ab^2 \\ &= \frac{3}{8ab^2}\int_\pi^0 r^4 \sin\theta\,d\theta = \frac{3a^3(1-e^2)^4}{8b^2}\int_\pi^0 \frac{\sin\theta\,d\theta}{(1-e\cos\theta)^4} \\ &= \frac{a(1-e^2)^3}{8e}\left\{\frac{1}{(1-e)^3} - \frac{1}{(1+e)^3}\right\} = \frac{a(1-e^2)^3}{8e}\left\{\frac{(1+e)^3-(1-e)^3}{(1-e^2)^3}\right\} \\ &= a\frac{3+e^2}{4}. \end{aligned}$$

13632. (Professor E. LEMOINE.)—Soit un triangle ABC, soit K un point de son plan ; AK, BK, CK coupent respectivement BC, CA, AB en α, β, γ. Par α, β, γ je mène des parallèles respectivement à CA, AB, BC qui coupent AB, BC, CA en ν, λ, μ et des parallèles à AB, BC, CA qui coupent CA, AB, BC en μ_1, ν_1, λ_1. On sait que Aλ, Bμ, Cν concourent en un point que j'appelle ω et Aλ_1, Bμ_1, Cν_1 en un point que j'appelle ω_1. (1) Déterminer K de façon que ω et ω_1 soient sur le cercle circonscrit à ABC. (2) Trouver le lieu de K lorsque la conique A, B, C, ω, ω_1 est une hyperbole équilatère. (3) Dans ce cas quel est le lieu du centre de cette hyperbole ?

Solution by the PROPOSER.

(1) K est le point dont les coordonnées normales sont

$$(a^4-b^2c^2)/a, \quad (b^4-c^2a^2)/b, \quad (c^4-a^2b^2)/c.$$

(2) Le lieu est la conique $\Sigma a\cos A\,(a^2x^2-bcyz) = 0$. Le centre de cette conique est sur la ligne qui joint le barycentre au centre du cercle circonscrit, il a pour coordonnées normales $(4S-3aR\cos A)/a$, ..., R étant le rayon du cercle circonscrit et S la surface de ABC.

(3) Le lieu est la conique $\Sigma\left[a^3x^2\cos A - bcyz\,(b\cos B + c\cos C)\right] = 0$.

13636. (Professor SANJÁNA.)—Prove that

$$\int_0^{\frac{1}{2}\pi}\int_0^1 \tan^{-1}\frac{2m\cos\theta}{1-m^2}\,dm\,d\theta = \tfrac{1}{4}\pi^2 - 2\log 2,$$

where m is less than unity.

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Solution by Rev. J. Cullen, B.A.

Let $$u = \int_0^{\frac{1}{2}\pi} \tan^{-1} \frac{2m \cos\theta}{1-m^2}\, d\theta;$$

$$\therefore \quad \frac{du}{dm} = 2(1+m^2) \int_0^{\frac{1}{2}\pi} \frac{\cos\theta\, d\theta}{(1+m^2)^2 - 4m^2 \sin^2\theta} = \frac{1}{m} \log \frac{1+m}{1-m}$$

$$= 2\left(1 + \frac{m^2}{3} + \frac{m^4}{5} + \frac{m^6}{7} + \ldots\right);$$

$$\therefore \quad u = 2\left(m + \frac{m^3}{3^2} + \frac{m^5}{5^2} + \frac{m^7}{7^2} + \ldots\right).$$

Hence $$I = \int_0^1 u\, dm = 2\left(\frac{1}{2} + \frac{1}{4 \cdot 3^2} + \frac{1}{6 \cdot 5^2} + \frac{1}{8 \cdot 7^2} + \ldots\right)$$

$$= 2\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots - \left(\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \ldots\right)\right]$$

$$= \tfrac{1}{4}\pi^2 - 2 \log 2.$$

13631. (The late Professor Sylvester, F.R.S. Jan. 31, 1897.)—Show, if ρ is any irrational p^{th} root of unity (p being a prime number), the sum of the $(p-1)$ values of $1/\{(1-\rho)(1-\rho^2)\}$ is equal to $\{(1-p)(p-11)\}/24$.

Solution by Rev. J. Cullen, B.A.

Since $\Sigma 1/(1-\rho)(1-\rho^2) = \frac{1}{2}\Sigma\left[1/(1-\rho)^2 + 1/(1-\rho^2)\right]$ and $x^p - 1 = 0$, we find by ordinary working that $1/(1-\rho)^2$ is a root of

$$x^{p-1} + \left\{\tfrac{1}{3}\left[(p-1)(p-2)\right] - \tfrac{1}{4}(p-1)^2\right\} x^{p-2} + \ldots = 0;$$

therefore $$\Sigma 1/(1-\rho)^2 = (1-p)(p-5)/12.$$

Similarly, $$\Sigma 1/(1-\rho^2) = (p-1)/2;$$

therefore $$\Sigma 1/(1-\rho)(1-\rho^2) = (1-p)(p-11)/24.$$

13598. (P. W. Flood.)—Bonnycastle, in his *Algebra*, requires two numbers in the proportion of 8 : 15 such that the sum of their squares will be a square. Why in the ratio of 8 : 15 ?

Solution by Lt.-Col. Allan Cunningham, R.E.

Taking two numbers mx, nx in the ratio $m : n$, here

$$(mx)^2 + (nx)^2 = (m^2 + n^2)\, x^2,$$

and, in order that this last may be a square, $(m^2 + n^2)$ must be a square. Now in an *elementary* question the numbers $m : n$ would probably be selected so as to give a solution in integers, *i.e.*, so that $m^2 + n^2 =$ integer square; and $8^2 + 15^2 = 17^2$ is *one* way of satisfying this.

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13688. (Professor A. Droz-Farny.)—Soit AB une corde quelconque d'une parabole P, parallèle à la tangente en un point fixe C de P. Par les points A et B on fait passer une circonférence tangente en D, à la tangente fixe du point C. Le diamètre du point D coupe AB en L et la circonférence en M. LM est égale au double du rayon de courbure de C.

Solution by H. W. Curjel, M.A.

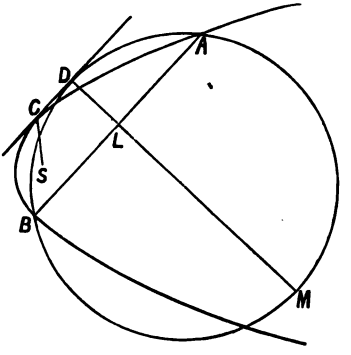
Let S be the focus.

Then

$$DL \cdot LM = BL^2$$
$$= 4CS \cdot CL;$$

but DL : CL is constant.

Therefore LM is constant, and is equal to the diameter of curvature at C; for, when AB coincides with the tangent at C, LM becomes the diameter of the circle of curvature at C.



13666. (Professor Thomas Savage, M.A.)—Given in magnitude the side AB of the triangle ABC, in position and magnitude the internal bisector AD of the angle BAC, and the centre of the circumcircle of the triangle ADC, to construct the triangle ABC.

Solution by Lionel E. Reay, B.A.

Locus of B is a circle with A as centre.

Locus of C is a circle through A and D.

For the triangle required, B, D, and C are collinear.

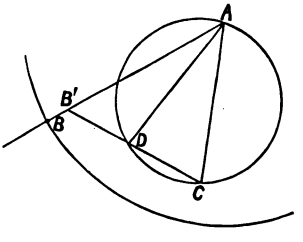
If C is any point on the locus, and B is such that $\angle BAD = \angle DAC$, and CD cuts AB in B′, then

$$\text{pencil A (B′)} = \text{A (C)} = \text{D (C)}$$
$$= \text{D (B′)};$$

therefore locus of B′ is a conic through A and D.

Taking any two other positions of B′, this conic is completely determined. A point where it intersects the locus of B obviously gives the angular point B of the triangle required.

In the figure, locus of B′ is a hyperbola, branches through D and A. The D branch gives two positions of B. For the A branch, AD would be external bisector.



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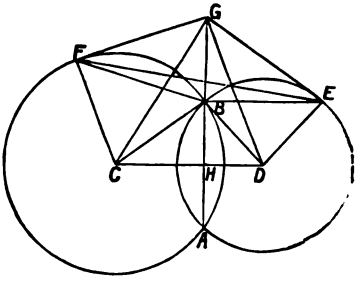
13544. (J. Macleod, M.A.)—Two circles whose centres are C, D respectively intersect in A, B, CB being greater than DB. From point G in AB produced the tangent GF is drawn to the larger circle, and tangent GE on the opposite side to the smaller. Prove that $\angle FCB$ is greater than, equal to, or less than $\angle BDE$ according as FE cuts GA between B and A, in B, or between B and G.

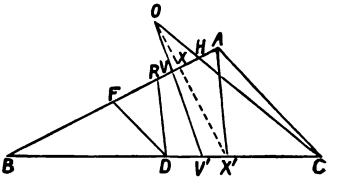
Solution by the Proposer.



In the diagram chosen, FE cuts AB between B and G. GC being greater than GD, $\angle CGH$ is greater than $\angle DGH$. By cutting off from FC a part equal to ED and joining with G, it is seen that $\angle FGC$ is greater than $\angle EGD$, and much more is $\angle FGB$ than $\angle EGB$; hence FB is greater than BE and $\angle FEB$ than $\angle EFB$. From which it follows that $\angle GEB$ is greater than $\angle GFB$, and therefore $\angle BDE$ is greater than $\angle FCB$.

13647. (I. Arnold.)—From a given point P outside a given triangle ABC to draw a line bisecting the triangle.

Solution by W. C. Stanham, B.A.



Let ABC be the triangle, O the given point, and BD = DC. Draw OHC, and draw DF parallel to CA. Let any line OXX′ be drawn; join AX′ and draw DR parallel to AX′. Then, since FDR and CAX′ are equal angles, FH is divided homographically by the corresponding points R, X. Draw OVV′ through a double point V of the system. Then, since DV is parallel to AV′,

triangle BVV′ = triangle BDA = $\frac{1}{2}$ABC.

13569. (Rev. J. Cullen, B.A.).—A triangle is formed by two co-initial vectors α, β and the join of their terms. Prove that $\alpha\Omega\beta^{-1}$ is the

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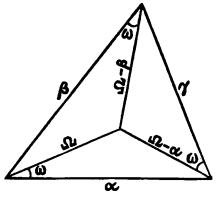
vector to the negative BROCARD-point, where Ω is vector to the positive point. Also express these vectors in terms of α and β.

Solution by the PROPOSER.

$\tan\omega = \mathrm{TV}\alpha\Omega/\mathrm{S}\alpha\Omega = \mathrm{TV}\beta(\Omega-\beta)/\mathrm{S}\beta(\Omega-\beta)$
$= \mathrm{TV}\gamma(\Omega-\alpha)/\mathrm{S}\gamma(\Omega-\alpha)$,
where $\gamma = \alpha-\beta$.
Assuming $\Omega = a\alpha+b\beta$, we find
$a = \beta^2\gamma^2/\Sigma\alpha^2\beta^2$, $b = \alpha^2\beta^2/\Sigma\alpha^2\beta^2$.
Ω_1 the negative BROCARD-point vector is obtained from Ω by interchanging α and β; therefore $\Omega_1 = (\alpha^2\gamma^2\beta+\alpha^2\beta^2\alpha)/\Sigma\alpha^2\beta^2$.



Hence $\Omega_1\beta = a\Omega$ or $\Omega_1 = a\Omega\beta^{-1}$. These vectors may be expressed in the more compact form, thus

$$\Omega^{-1} = \beta^{-1}+\gamma^2\beta^{-2}\alpha^{-1}, \quad \Omega_1^{-1} = \alpha^{-1}+\gamma^2\alpha^{-2}\beta^{-1}.$$

13231. (Professor FINKEL.)—A pole, a certain length of whose top is painted white, is standing on the side of a hill. A person at A observes that the white part of the pole subtends an angle equal to α, and on walking to B, a distance a, directly down the hill towards the foot of the pole, the white part subtends the same angle. What is the length of the white part, if the point B is at a distance b from the foot of the pole?

Solution by K. S. PUTNAM.

Let CD be the pole, DE the painted part, A and B the points of observation.

Since angle DAE = DBE, the points A, B, D, and E are concyclic.

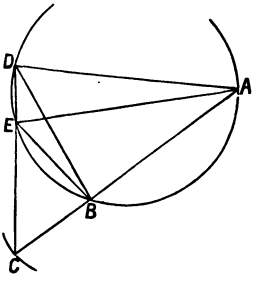
The secants from C give

$$\mathrm{AC}\,.\,\mathrm{CB} = \mathrm{DC}\,.\,\mathrm{CE}.$$

If the pole is a certain length, $= l$,

we have $l(l-\mathrm{DE}) = b(a+b)$;

$$\therefore \quad \mathrm{DE} = [l^2-b(a+b)]/l.$$



If the white part is a certain portion of the pole, as $1/n^{\text{th}}$,

$$n\,.\,\mathrm{DE}(n-1)\,\mathrm{DE} = b(a+b); \qquad \therefore \quad \mathrm{DE} = \left[\{b(a+b)\}/\{n(n-1)\}\right]^{\frac{1}{2}}.$$

[For another Solution of this Question, see Vol. LXVI., p. 91.

The present solution takes no notice of the angular value α, nor of the gradient of the hill, which we may surely take as $\sin\beta$. If from O, the

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centre of the circle, a parallel and a perpendicular to AC be drawn, meeting DE (produced) in P and Q respectively, QB will divide OP in the ratio $\frac{1}{2}a : b$; we also have $r = \frac{1}{2}DE \cdot \operatorname{cosec} \alpha$,

whilst $OP = \frac{1}{2}DE \cdot \cot \alpha \sec \beta$, and $OQ = \frac{1}{2}DE \cot \alpha \operatorname{cosec} \beta$.

With these data, we easily find $DE = (a + 2b) \sin \alpha \cos \beta / (\cos \alpha \pm \sin \beta)$].

13531. (PHILIPPA G. FAWCETT.)—Prove the following rule for finding the power of the lens equivalent to n thin lenses arranged along an axis:—Take any combination out of the n lenses, and form the continued product of the powers and the distances between consecutive lenses of the combination as they stand; the sum of the products so formed for all possible combinations out of the n lenses is equal to the power of the equivalent lens. (If the combination consists of one lens only, the corresponding term is its power simply.) Extend the rule to any optical instrument symmetrical about an axis.

Solution by Professor SANJÁNA, M.A.

Suppose the focal lengths $f_1, f_2, \ldots, f_n$; the intervals $a_1, a_2, \ldots, a_{n-1}$. Let a ray parallel to the common axis strike the lenses respectively at distances $y_1, y_2, \ldots, y_n$ from the axis, and let $\partial_1, \partial_2, \ldots, \partial_n$ be the total deviations at the several lenses. Then ∂_n is the numerator of the last convergent in $\dfrac{y_1}{1+}\dfrac{1}{k_1+}\dfrac{1}{a_1+}\dfrac{1}{k_2+}\cdots\dfrac{1}{a_{n-1}+}\dfrac{1}{k_n}$, where $k_1, k_2, \ldots, k_n$ are the reciprocals of the focal lengths (HEATH, *Optics*, § 83). If F_n be the focal length of the equivalent lens, $1/F_n = \partial_n/y_1$; thus $1/F_n$ is the numerator of the last convergent in

$$\frac{1}{1+}\frac{1}{k_1+}\frac{1}{a_1+}\frac{1}{k_2+}\cdots\frac{1}{+k_n}.$$

The values of $1/F_1$, $1/F_2$, $1/F_3$, $1/F_4$, ... are, therefore,

k_1; $a_1k_1k_2 + k_2 + k_1$; $a_1a_2k_1k_2k_3 + a_2k_3k_1 + a_2k_3k_2 + a_1k_1k_2 + a_1k_1k_3 + k_1 + k_2 + k_3$;

$a_1a_2a_3k_1k_2k_3k_4 + a_1a_2k_1k_2k_3 + a_1a_3k_1k_2k_4 + a_1a_3k_1k_3k_4 + a_2a_3k_1k_3k_4 + a_1a_2k_1k_2k_4$
$+ a_2a_3k_2k_3k_4 + a_3k_1k_4 + a_3k_2k_4 + a_3k_3k_4 + a_1k_1k_2 + a_1k_1k_3 + \ldots + k_1 + k_2 + k_3 + k_4$.

By writing these in the following way, the rule mentioned becomes obvious:—

k_1; $k_1k_2 \cdot a_1 + k_1 + k_2$;

$k_1k_2k_3 \cdot a_1a_2 + k_1k_2 \cdot a_1 + k_1k_3(a_1 + a_2) + k_2k_3 \cdot a_2 + k_1 + k_2 + k_3$;

$k_1k_2k_3k_4 \cdot a_1a_2a_3 + k_1k_2k_3 \cdot a_1a_2 + k_1k_3k_4(a_1 + a_2)a_3 + k_1k_2k_4(a_2 + a_3)a_1$
$+ k_2k_3k_4 \cdot a_2a_3 + k_1k_2 \cdot a_1 + k_1k_3(a_1 + a_2) + k_1k_4(a_1 + a_2 + a_3)$
$+ k_2k_3 \cdot a_2 + k_2k_4(a_2 + a_3) + k_3k_4 \cdot a_3 + k_1 + k_2 + k_3 + k_4$.

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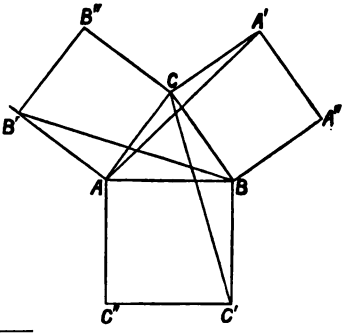
13568. (W. S. Cooney, B.A.)—Squares are described on sides of any triangle ABC (Extension of Quest. 13481). If T = area of triangle formed by intersections of AA′, BB′, CC′, T′ = area of triangle formed by the whole lines AA′, BB′, CC′, and Δ = area of ABC, prove geometrically that $TT' = \Delta^2$.



Solutions by (1) *the* Proposer; (2) *by* Professor Sanjána, M.A.

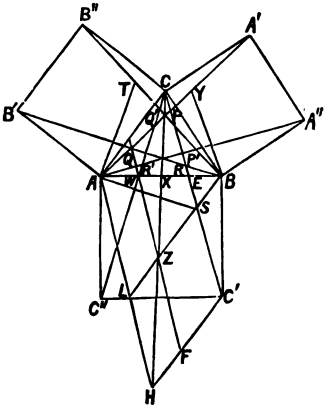
(1) Let CC′, AA′ intersect in P, AA′, BB′ in Q, BB′, CC′ in R, AA″, BB″ in P′, BB″, CC″ in Q′, CC″, AA″ in R′. Complete parallelogram ACC′H. Let CC″, CH, CC′ meet AB in W, X, E respectively. Draw BL parallel to AC, meeting CH in Z, and AH in L. Then Q, W, Z are collinear and parallel to CC′;* therefore triangle ABC

or $\Delta = \frac{1}{2}$ parallelogram CL,

$= \frac{1}{2}$ parallelogram CF,

= triangle CQC′.

Again, if parallelogram B′AC″α be completed, it is evident triangle CC′α is triangle formed by AA′, BB′, CC′ or T′, and is similar to PQR or T; therefore



$$T/T' = (PR)^2/(CC')^2,$$

but $$T/CQC' = T/\Delta = PR/CC';$$

therefore $T/T' = T^2/\Delta^2$; therefore $TT' = \Delta^2$.

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* $\Delta TAP' = \Delta ASP$, since AT = AS (Quest. 13481); therefore TP′ = AP. But PQ = P′Q′; therefore TQ′ = AQ. Similarly, PY = BR; therefore AY = BT, and AY and BT are similarly divided; therefore AW/WB = TQ′/Q′B = AQ/QY; therefore QW is parallel to BY or to CC′. Now it is easy to prove that AX/XB = EX/XW; therefore BZ parallel to AC, and WZ parallel to CC′, meet CH at the same point Z.

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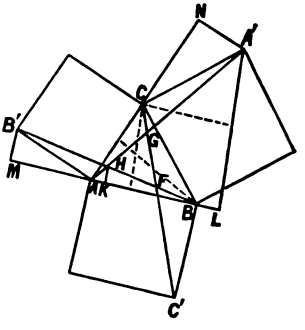
(2) The following trigonometrical solution is independent of the properties in Quest. 13481.



Drawing HK, A'L, B'M perpendiculars to AB,

$$\frac{AK}{HK} = \frac{AL}{LA'} = \frac{b\cos A + b\sin A}{a\sin B + a\cos B}$$
$$= \frac{1+\cot A}{1+\cot B};$$
$$\frac{BK}{HK} = \frac{c + b\sin A}{b\cos A}$$
$$= \frac{1+\cot A+\cot B}{\cot A};$$

therefore $\frac{AK+BK}{HK}$,

i.e., $$\frac{c}{HK} = \frac{\cot^2 A+\cot^2 B+2\cot A+2\cot B+\cot A\cot B+1}{\cot A\,(1+\cot B)}$$
$$= \frac{(\cot A+\cot B)^2+2(\cot A+\cot B)+(1-\cot A\cot B)}{\cot A\,(1+\cot B)}$$
$$= \frac{(\cot A+\cot B)^2+2(\cot A+\cot B)+\cot C(\cot A+\cot B)}{\cot A\,(1+\cot B)}.$$

Hence $$\frac{c^2}{c\,.\,HK(\cot A+\cot B)} = \frac{\cot A+\cot B+\cot C+2}{\cot A\,(1+\cot B)},$$

i.e., $$\frac{2\Delta}{2HAB} = \frac{\cot\omega+2}{\cot A\,(1+\cot B)}.$$

Similarly the areas FBC, GCA are found. Hence

$$\frac{T}{\Delta} = 1-\frac{\cot A+\cot A\cot B}{\cot\omega+2}-\frac{\cot B+\cot B\cot C}{\cot\omega+2}-\frac{\cot C+\cot C\cot A}{\cot\omega+2}$$
$$= 1-\frac{\cot\omega+1}{\cot\omega+2} = \frac{1}{\cot\omega+2}.$$

Also $AA'^2 \equiv a^2 = AC^2+CA'^2+2AC\,.\,CN = a^2+b^2+4\Delta$;

so for β^2 and γ^2. Hence

$$16T'^2 = 2\beta^2\gamma^2+2\gamma^2a^2+2a^2\beta^2-\alpha^4-\beta^4-\gamma^4$$
$$= 4b^2c^2+4c^2a^2+4a^2b^2+16a^2\Delta+16b^2\Delta+16c^2\Delta+48\Delta^2$$
$$= 16\Delta^2\operatorname{cosec}^2\omega+64\Delta^2\cot\omega+48\Delta^2$$
$$= 16\Delta^2\cot^2\omega+64\Delta^2\cot\omega+64\Delta^2;$$

therefore $T' = \Delta(\cot\omega+2)$, and $TT' = \Delta^2$.

13698. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Sum, to n terms, the series $0+4+18+48+100+180+294+\ldots$.

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Solution by G. H. WEBB; H. W. CURJEL, M.A.; *and many others.*

$$\begin{aligned}\text{Sum} &= (1^3-1^2)+(2^3-2^2)+(3^3-3^2)+(4^3-4^2)+\ldots+(n^3-n^2)\\ &= \{1^3+2^3+3^3+4^3+\ldots+n^3\}-\{1^2+2^2+3^2+4^2+\ldots+n^2\}\\ &= \left[\tfrac{1}{2}\{n(n+1)\}\right]^2-\tfrac{1}{6}\{n(n+1)(2n+1)\}\\ &= \tfrac{1}{12}\{(n-1)n(n+1)(3n+2)\}.\end{aligned}$$

13630. (Professor CROFTON, F.R.S.)—Show that

$$1-4+4^2-4^3+\ldots+4^{2r}$$

is the sum of two squares; and give their values, according as r is odd or even.

Solution by H. W. CURJEL, M.A.; *and* Rev. J. CULLEN, B.A.

$$\text{The given series} = \frac{1+(2^{2r+1})^2}{5} = \left\{\frac{2(4^r\pm1)}{5}\right\}^2+\left(\frac{4^{r+1}\mp1}{5}\right)^2,$$

the upper sign being taken if r is odd, and the lower if R is even.

12680. (Professor SHIELDS.)—Let n be any number, and let d be the difference; then

$$(n^3+n^3)/(nn+d^2) = (n+n), \quad [(n^3+n)+(n^3+n)]/(nn+d^2+1) = (n+n),$$
$$(n^5+n^5)/[(nn+d^2)^2+(nnd^2)] = (n+n).$$

Solution by Rev. J. L. KITCHIN, M.A.

I take the notation, in this question, as being either numerical, algebraical, or functional; but that the n's, as they stand, are not identical.

To be clear, let x stand for the first n, y for the second; then $d=x-y$,

$$\frac{x^3+y^3}{xy+(x-y)^2} = \frac{x^3+y^3}{x^2-xy+y^2} = x+y \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or

$$(x^3+x^3)/(xn+d^2) = (x+n),$$

$$\frac{(n^3+n)+(n^3+n)}{nn+d^2+1} = \frac{(x^2-xy+y^2+1)(x+y)}{x^2-xy+y^2+1} = (x+y) = (x+n)\ldots(2),$$

$$\frac{n^5+n^5}{(nn+d^2)^2+(nnd^2)^2} = \frac{x^5+y^5}{x^4-x^3y+x^2y^2-xy^3+y^4} = x+y \quad \ldots\ldots\ldots(3);$$

therefore

$$(x^5+n^5)/\{(nn+d^2)^2+nnd^2\} = (n+n).$$

Of course we might deal in the same way with (n^7+n^7), &c., &c., and get corresponding results.

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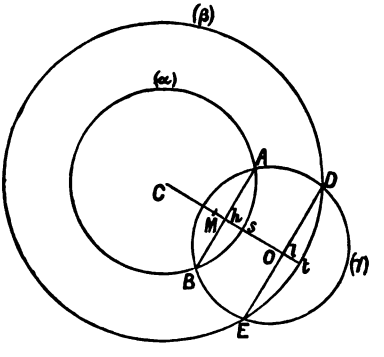
13417. (MORGAN BRIERLEY.)—From a point midway in the space between two concentric circles describe another to intersect them; the versed sines of the intercepted arcs adjacent to the point are equal to each other. Required the demonstration.

Solution by W. E. JEFFARES, B.A.



Let C be the centre of the concentric circles (α), (β) and O that of the arbitrary circle (γ) whose centre is midway between them; r, r', R their radii, and M the middle point of CO. Since AB, the chord of intersection, is the radical axis of (α) and (γ), we have

(1) $C\hbar^2 - O\hbar^2 = r^2 - R^2$,

or $2CO \times \hbar M = r^2 - R^2$.

(2) Similarly

$2CO \times lM = r_1^2 - R^2$.

If we take the difference of (1) and (2), we have

$2CO \times \hbar l = r_1^2 - r^2$;

but $CO = \frac{1}{2}(r + r')$; therefore $l\hbar = r' - r$, $r' - \hbar l = r' - (r' - r)$, or $C\hbar + lt = r$. Again $C\hbar + \hbar S = r$; therefore $lt = \hbar S$.

Hence the versed sines of the intercepted arcs are equal.

13646. (ROBERT W. D. CHRISTIE.)—Every triangular equals the sum of three others, and therefore every integer equals the sum of $2n+1$ triangulars.

Solution by the PROPOSER.

$$T_{3a+1} \equiv T_{2a+1} + T_{2a} + T_a, \qquad T_{3a+2} \equiv T_{2a+1} + T_{2a+1} + T_{a+1},$$
$$T_{3a+3} \equiv T_{2a+2} + T_{2a+2} + T_a;$$

and, since FERMAT's theorem that every integer equals the sum of three triangulars has been proved by CAUCHY, the theorem is proved.

13653 & 13684. (J. C. WASSON.)—A, B, C are the vertices of a triangle inscribed in a circle. A′, B′, C′ are three other points on the circle, such that arc AA′ + arc BB′ + arc CC′ = 0 or a multiple of 2π (the

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algebraic sum of the arcs being taken). Then the SIMSON-lines of the points A′, B′, C′, with respect to the triangle ABC, meet in a point.

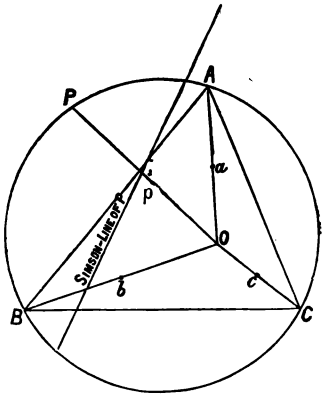
(13684.) (J. C. WASSON.)—ABC is a triangle inscribed in a circle, and α, β, γ are three points on the circle such that

$$\text{arc } A\alpha + \text{arc } B\beta + \text{arc } C\gamma = (2n+1)\pi.$$

The nine-point circle of the triangle formed by the SIMSON-lines corresponding to α, β, γ is the nine-point circle of the original triangle ABC.

Solutions by W. J. DOBBS, M.A.

13653. Let O be the orthocentre and P any point on the circumcircle of triangle ABC, and let p be the middle point of OP. Then, as P describes the circumcircle, p describes the nine-point circle. We will speak of p as the point of the nine-point circle corresponding to P; also the points corresponding to A, B, ... are denoted by a, b, ..., respectively.



That portion of the SIMSON-line of P which is intercepted between BC and OA is bisected at p. Hence it follows, as in Quest. 13628, that the motion of the SIMSON-line is identical with that of a chord of the nine-point circle one extremity of which passes over twice the arc described by the other, the arcs being measured in opposite directions.

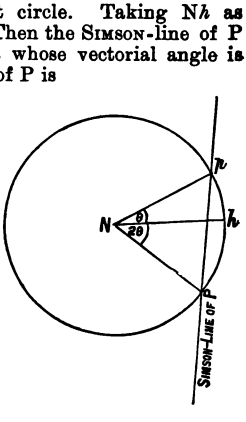
Now let h be one of the three points where a SIMSON-line touches the nine-point circle, and N the centre of that circle. Taking Nh as initial line, let θ be the vectorial angle of p. Then the SIMSON-line of P meets the nine-point circle again at the point whose vectorial angle is -2θ. Hence the equation to the SIMSON-line of P is

$$x\cos\tfrac{1}{2}\theta - y\sin\tfrac{1}{2}\theta = \rho\cos\tfrac{3}{2}\theta,$$

where ρ is the radius of the nine-point circle.

This equation can be written in the form:

$$y\tan^3\tfrac{1}{2}\theta - (x+3\rho)\tan^2\tfrac{1}{2}\theta + y\tan\tfrac{1}{2}\theta + \rho - x = 0.$$



If x and y be supposed known, we have here a cubic to determine $\tan\frac{1}{2}\theta$. This gives *six* different geometrical values of $\frac{1}{2}\theta$, but *three* different geometrical values of θ, giving three different positions of p and therefore three different positions of P.

Let the SIMSON-lines of A′, B′, C′ meet at the point (x, y). Then, if α', β', γ' are the vectorial angles of a', b', c', respectively,

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we must have $\tan\frac{1}{2}\alpha' + \tan\frac{1}{2}\beta' + \tan\frac{1}{2}\gamma' = (x+3\rho)/y$;

$$\tan\tfrac{1}{2}\beta'\tan\tfrac{1}{2}\gamma' + \tan\tfrac{1}{2}\gamma'\tan\tfrac{1}{2}\alpha' + \tan\tfrac{1}{2}\alpha'\tan\tfrac{1}{2}\beta' = 1;$$

$$\tan\tfrac{1}{2}\alpha'\tan\tfrac{1}{2}\beta'\tan\tfrac{1}{2}\gamma' = (x-\rho)/y.$$

The second equation gives the necessary and sufficient connexion between α', β', γ'; the two other equations going to determine x and y.

Hence the Simson-lines of A', B', C' will meet in a point, provided

$$\frac{\tan\frac{1}{2}\beta' + \tan\frac{1}{2}\gamma'}{1-\tan\frac{1}{2}\beta'\tan\frac{1}{2}\gamma'} = \cot\tfrac{1}{2}\alpha';$$

i.e., provided $\frac{1}{2}(\beta'+\gamma') = n\pi + \frac{1}{2}\pi - \frac{1}{2}\alpha'$;

i.e., provided $\alpha'+\beta'+\gamma' = (2n+1)\pi$ ..... (1).

Now let α, β, γ be the vectorial angles of a, b, c. Then, as the Simson-lines of A, B, C meet at a point, namely, the orthocentre of the triangle, we have $\alpha+\beta+\gamma = (2m+1)\pi$.

Hence the condition (1) can be put in the form

$$\alpha'-\alpha+\beta'-\beta+\gamma'-\gamma = 0 \text{ or a multiple of } 2\pi.$$

Thus, we see that the Simson-lines of A', B', C' will meet in a point, provided arc AA' + arc BB' + arc $CC' = 0$ or a multiple of 2π.

[The Proposer adds that the Simson-lines of A, B, C with respect to A'B'C' will also meet in this point.]

13684. Referring to my solution of Quest. 13653, suppose that α', β', γ' are connected by the relation

$$\alpha'+\beta'+\gamma' = 2n\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Then the equation to $b'c'$ is

$$x\cos\tfrac{1}{2}(\beta'+\gamma') + y\sin\tfrac{1}{2}(\beta'+\gamma') = \rho\cos\tfrac{1}{2}(\beta'-\gamma'),$$

and this is parallel to

$$x\cot\tfrac{1}{2}(\beta'+\gamma') + y = 0, \quad i.e., \text{ to } \quad -x\cot\tfrac{1}{2}\alpha' + y = 0;$$

therefore parallel to $x\cos\frac{1}{2}\alpha - y\sin\frac{1}{2}\alpha = \rho\cos\frac{3}{2}\alpha$,

i.e., parallel to the Simson-line of A'.

In the same way, $c'a'$ and $a'b'$ are respectively parallel to the Simson-lines of B' and C'. This evidently requires that the nine-point circle of the original triangle must be the nine-point circle of the triangle formed by the Simson-lines of A', B', C'.

But, as $\alpha+\beta+\gamma = (2m+1)\pi$, the condition (2) may be written thus:

$$\alpha'-\alpha+\beta'-\beta+\gamma'-\gamma = (2r+1)\pi.$$

Thus, if arc AA' + arc BB' + arc $CC' = (2r+1)\pi$,

the nine-point circle of the triangle formed by the Simson-lines of A', B', C' coincides with the nine-point circle of the original triangle.

11412. (Professor Tissot.)—Construire un cercle tangent à une droite donnée et bitangent à une conique donnée.

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Solution by H. W. Curjel, M.A.

Let the foci of the given conic be S and S′, and excentricity = e. Draw a conic, excentricity e, with focus S and directrix the given straight line l. Then the circles with centres G_1, G_2, G_3, G_4, where this conic cuts the axes of the given conic, touching l will touch the given conic each in two points. For, if the normal to a conic at P cuts an axis in G,

$$GS : GP = e : 1.$$

12941 & 13652. (Professor Gruber.)—Find the first six integral values of n in $\frac{1}{2}n(n+1) = \square$.

(13652.) (J. Woodcote West.) — Sum the series $\square_1 + \square_2 + \square_3 + \ldots$ to n terms, where $\square_n$ represents the nth square number of the form $\frac{1}{2}\{a(a+1)\}$.

Solution by Professor Sanjána, M.A.; *and others.*

Let $\frac{1}{2}n(n+1) = y^2$; then

$$n^2 + n - 2y^2 = 0 \quad \text{and} \quad n = \tfrac{1}{2}\left\{-1 + (8y^2+1)^{\frac{1}{2}}\right\}.$$

Thus $$8y^2 + 1 \equiv x^2 \quad \text{and} \quad x^2 - 8y^2 = 1.$$

One solution of this is $x = 3$, $y = 1$; so that $x^2 - 8y^2 = (3^2 - 8 \cdot 1^2)^n$. Hence we may assume

$$x + 2\sqrt{2}\,y = (3+2\sqrt{2})^n, \quad x - 2\sqrt{2}\,y = (3-2\sqrt{2})^n.$$

These give $x = \frac{1}{2}\left\{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n\right\}$ and $n = \frac{1}{2}(-1+x)$;

therefore $$n = \tfrac{1}{4}\left\{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n - 2\right\},$$

Putting n (in the index) equal to 1, 2, 3, 4, 5, ..., we get the integral values required equal to 1, 8, 49, 288, 1681, ..., the corresponding values of y^2 being 1^2, 6^2, 35^2, 204^2, 1189^2, ... (12941).

To find the sum of the first n values of y^2, we observe that

$$y^2 = \tfrac{1}{8}(x^2-1) = \tfrac{1}{8}\left[\tfrac{1}{4}\left\{(3+2\sqrt{2})^{2n} + (3-2\sqrt{2})^{2n} - 2\right\}\right].$$

Putting $a = 3+2\sqrt{2}$, $b = 3-2\sqrt{2}$, we see that the sum required

$$= \tfrac{1}{32}\Sigma_1^n(a^{2n} - 2 + b^{2n}) = \tfrac{1}{32}\left\{\frac{a^2(a^{2n}-1)}{a^2-1} - 2 + \frac{b^2(1-b^{2n})}{1-b^2}\right\}. \quad (13652).$$

[For another solution of Question 12941, see Vol. lxv., p. 53.]

13500. (Professor Ramaswami Aiyar, M.A.)—The natural harmonic progression being divided into the two series

$$1 + \tfrac{1}{3} + \tfrac{1}{5} + \ldots \quad \text{and} \quad \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \ldots,$$

show that the sum of n terms of the former series is less than the sum of $4n$ terms of the latter series by a quantity which diminishes as n increases and vanishes when $n = \infty$.

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Solution by H. W. CURJEL, M.A.

$$\frac{1}{8n-6}+\frac{1}{8n-4}+\frac{1}{8n-2}+\frac{1}{8n}-\frac{1}{2n-1}=\frac{-16n^2+20n-3}{4n\,(4n-1)\,(4n-2)\,(4n-3)};$$

which is positive if $n = 1$ and negative if $r =$ or > 2.

Hence the theorem follows if the sum to infinity of the series formed by subtracting one term of the former series from four of the latter in order is zero, which is clear, since this sum = limit when $x = 1$ of

$$\tfrac{1}{2}x+\tfrac{1}{4}x^2+\tfrac{1}{6}x^3+\tfrac{1}{8}x^4+\ldots-\tfrac{1}{3}x^4-\tfrac{1}{6}x^6-\tfrac{1}{8}x^{10}-\ldots;$$

i.e., of $$-\tfrac{1}{2}\log(1-x)-\tfrac{1}{2}\log\left\{(1+x^2)/(1-x^2)\right\};$$

i.e., of $$\tfrac{1}{2}\log\left\{(1+x)/(1+x^2)\right\}=0.$$

13566. (W. E. JEFFARES, B.A.)—If α, β, γ, δ be the roots of the equation $ax^4+4bx^3+6cx^2+4dx+c=0$, find the relation between the coefficients so that $\alpha-\beta$, $\alpha-\gamma$, $\alpha-\delta$ may be in harmonic progression.

Solution by Rev. J. CULLEN, B.A.; *and the* PROPOSER.

The relation $$1/(\alpha-\beta)+1/(\alpha-\delta)=2/(\alpha-\gamma)$$

gives $$\Sigma\alpha\beta=3(\alpha\gamma+\beta\delta)=6c/a;$$

therefore $$(\alpha+\gamma)(\beta+\delta)=4c/a.$$

Also $$\alpha+\gamma+\beta+\delta=-4b/a;$$

hence $$4c/a+4b(\alpha+\gamma)/a+(\alpha+\gamma)^2=0.$$

Again, since $\alpha\beta\gamma\delta = e/a$, we have

$$\alpha\gamma-\beta\delta=2(c^2-ae)^{\frac{1}{2}}/a;$$

therefore $$\alpha\gamma=\left\{c+(c^2-ae)^{\frac{1}{2}}\right\}/a,\quad \beta\delta=\left\{c-(c^2-ae)^{\frac{1}{2}}\right\}/a,$$

also $$\alpha\gamma(\beta+\delta)+\beta\delta(\alpha+\gamma)=-4d/a;$$

therefore $$(\alpha+\gamma)a(c^2-ae)^{\frac{1}{2}}=2\left[ad-b\left\{c+(c^2-ae)^{\frac{1}{2}}\right\}\right].$$

Whence, eliminating $(\alpha+\gamma)$, we have

$$(ac-b^2)(c^2-ae)+(ad-bc)^2=0.$$

13651. (R. F. MUIRHEAD.)—In the array

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{matrix}$$

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there are 40 determinants which can be formed by suppressing 2 columns and 1 row. Show that six of these can be chosen such that their vanishing involves the vanishing of all the others. Extend this to an array of p columns and q rows, and determinants of the order r, where r is less than either p or q.

Solution by LIONEL E. REAY, B.A.

Consider any three rows of the given array. From these can be formed $\frac{5!}{3!\,2!} = 10$ determinants by striking out two columns.

The three rows may be taken in $\frac{4!}{3!\,1!} = 4$ ways.

Hence in all there are 40 determinants.

Suppose the following vanish:—$(a_1b_2c_3)$, (1); $(b_1c_2d_3)$, (2); $(c_1d_2e_3)$, (3); $(a_2b_3c_4)$, (4); $(b_2c_3d_4)$, (5); $(c_2d_3e_4)$, (6).

Consider *rows* (1), (2), and (3). Determinants (1) and (2) may be written

$$a_1\mathrm{K} + a_2\mathrm{L} + a_3\mathrm{M} = 0 \quad \text{and} \quad d_1\mathrm{K} + d_2\mathrm{L} + d_3\mathrm{M} = 0;$$

also $$b_1\mathrm{K} + b_2\mathrm{L} + b_3\mathrm{M} \equiv 0, \quad c_1\mathrm{K} + c_2\mathrm{L} + c_3\mathrm{M} \equiv 0;$$

therefore we have $$(a_1b_2d_3) = 0, \quad (a_1c_2d_3) = 0.$$

Similarly, being given also that $(c_1d_2e_3) = 0$, we can prove that all the determinants derived from the rows (1), (2), and (3) will vanish. Also the determinants from rows (2), (3), and (4) will vanish since determinants (4), (5), and (6) equal zero.

Consider now *columns* (1), (2), and (3). We are given

$$(a_1b_2c_3) = 0, \quad (a_2b_3c_4) = 0.$$

These may be written

$$a_1\mathrm{F} + b_1\mathrm{G} + c_1\mathrm{H} = 0, \quad a_4\mathrm{F} + b_4\mathrm{G} + c_4\mathrm{H} = 0;$$

also $$a_2\mathrm{F} + b_2\mathrm{G} + c_2\mathrm{H} \equiv 0, \quad a_3\mathrm{F} + b_3\mathrm{G} + c_3\mathrm{H} \equiv 0.$$

Hence $$(a_1b_2c_4) = 0 \quad \text{and} \quad (a_1b_3c_4) = 0.$$

Therefore, if two of the determinants vanish, all four vanish also. Therefore, combining the two results, if the six determinants given above vanish, all forty vanish also.

Again, considering the columns $\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \cdots & \cdots & \cdots \end{matrix}$ and writing (123) for the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, we see from above that for four rows given (123) = 0, (234) = 0, all other determinants vanish; and, for five rows given (123) = 0, (234) = 0, (345) = 0, the other determinants vanish.

Similarly, for six rows, there are 10 additional determinants—namely, (126), (136), (236), &c. Each of these vanishes if also (456) = 0; and so on. Therefore, for p rows, if the $p-2$ determinants (123), (234), (345), ..., $(p-2, p-1, p)$ vanish, all the other determinants vanish also.

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Similarly, if the determinants are of order 4, it may be shown that the vanishing of $p-3$ determinants makes the rest vanish. Hence, for determinants of order r, it is necessary that $(p-r+1)$ determinants vanish.

The same results apply for rows. Hence, for an array of p columns and q rows, there can be formed $\frac{p!}{r!\,(p-r)!}\,\frac{q!}{r!\,(q-r)!}$ determinants of order r, and there can be chosen from these $(p-r+1)(q-r+1)$ such that, if they vanish, the rest vanish also.

13682. (R. W. D. Christie.)—A father leaves a fortune of £46,189 to his four sons. The eldest gets a number of elevenths, the second thirteenths, the third seventeenths, and the fourth nineteenths, all the amounts being integral. The balance of £2,151 was given to charities. Find the amounts.

Solution by Rev. T. Roach, M.A.; *and* H. W. Curjel, M.A.

We find that $46,189 = 11 \times 13 \times 17 \times 19$.

Let number of elevenths, &c. $= a, b, c, d$; then

$$13 \times 17 \times 19a + 11 \times 17 \times 19b + \ldots = 46,189 - 2,151 = 44,038.$$

Divide by 11; then $8a-5$ is divisible by 11; therefore $a = 2$;

therefore $17 \times 19b + 13 \times 19c + 13 \times 17d = 3240$.

Divide by 13; then $11b-3$ is divisible by 13; therefore $b = 5$;

therefore $19c + 17d = 125$; $\therefore \quad c = 3, \quad d = 4$.

Therefore shares are 8,398; 17,765; 8,151; 9,724.

13681. (J. Woodcote West.)—If

$$\epsilon^{\int dy\,(P/M)} \quad \text{and} \quad \epsilon^{-\int dx\,(P/N)}$$

are both multipliers which render $M\,dx + N\,dy = 0$ integrable, then

$$\epsilon^{n\int dy\,(P/M)+(n-1)\int dx\,(P/N)}$$

is also such a multiplier.

Solutions (1) *by* Rev. J. Cullen, B.A.; (2) *by* H. W. Curjel, M.A.

(1) In general, if μ and μ' are two integrating factors of $M\,dx + N\,dy = 0$, they satisfy the equations

$$N\frac{d\mu}{dx} - M\frac{d\mu}{dy} = \left(\frac{dM}{dy} - \frac{dN}{dx}\right)\mu \quad \ldots\ldots\ldots\ldots \text{(i.)},$$

$$N\frac{d\mu'}{dx} - M\frac{d\mu'}{dy} = \left(\frac{dM}{dy} - \frac{dN}{dx}\right)\mu' \quad \ldots\ldots\ldots\ldots \text{(ii.)}.$$

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Also (i.) and (ii.) are satisfied by $\mu'' = \mu' \phi(\mu/\mu')$, where ϕ is an arbitrary function.

For multiply (i.) by $\phi'(\mu/\mu')$ and (ii.) by $\phi(\mu/\mu') - (\mu/\mu')\,\phi'(\mu/\mu')$; then we get, on adding,

$$\mathrm{N}\frac{d\mu''}{dx} - \mathrm{M}\frac{d\mu''}{dy} = \left(\frac{d\mathrm{M}}{dy} - \frac{d\mathrm{N}}{dx}\right)\mu''.$$

Hence the result is obtained by putting

$$\mu = e^{\int dy\,(\mathrm{P/M})}, \quad \mu' = e^{-\int dx\,(\mathrm{P/N})}, \quad \text{and} \quad \phi(\mu/\mu') = (\mu/\mu')^n.$$

(2) Let $\int e^{\phi}(\mathrm{M}\,dx + \mathrm{N}\,dy) = \mathrm{U}e^{\phi}$ and $\int e^{\psi}(\mathrm{M}\,dx + \mathrm{N}\,dy) = \mathrm{V}e^{\psi}$.

Then
$$(d/dx)(\mathrm{U}+\phi) = \mathrm{M} = (d/dx)(\mathrm{V}+\psi),$$
$$(d/dy)(\mathrm{U}+\phi) = \mathrm{N} = (d/dy)(\mathrm{V}+\psi);$$

therefore
$$(d/dx)\{n\mathrm{U} - (n-1)\,\mathrm{V} + n\phi - (n-1)\,\psi\} = \mathrm{M},$$
$$(d/dy)\{n\mathrm{U} - (n-1)\,\mathrm{V} + n\phi - (n-1)\,\psi\} = \mathrm{N};$$

therefore
$$\int e^{n\phi-(n-1)\psi}(\mathrm{M}\,dx + \mathrm{N}\,dy) = \{n\mathrm{U} - (n-1)\,\mathrm{V}\}\,e^{n\phi-(n-1)\psi}.$$

Putting
$$\phi = \int dy\,(\mathrm{P/M}) \quad \text{and} \quad \psi = -\int dx\,(\mathrm{P/N}),$$

we get the required result.

13695. (Professor Burnside, M.A.)—Find the differential equation of all concentric conics.

Solutions (1) *by* Rev. J. Cullen, B.A.; (2) *by* H. W. Curjel, M.A.

(1) All concentric conics are included in the equation

$$ax^2 + 2hxy + by^2 + 1 = 0;$$

whence, eliminating a, h, and b, we get

$$3xq^2 = r(xp - y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where
$$p = dy/dx, \quad q = d^2y/dx^2, \quad r = d^3y/dx^3.$$

(1) may also be expressed in the form

$$d/dx\left\{q\,(xp-y)^{-3}\right\}^m = 0,$$

since
$$q = \Delta/(hx+by)^3 \quad \text{and} \quad (hx+by)(xp-y) = 1.$$

(2) Differentiating the equation $ax^2 + 2hxy + by^2 = c$ three times, we

$$xh\,(y + xy_1) + hyy_1 = 0, \quad a + h\,(2y_1 + xy_2) + b\,(y_1^2 + yy_2) = 0,$$
$$0 + h\,(3y_2 + xy_3) + b\,(3y_1y_2 + yy_3) = 0.$$

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Hence required differential equation is

$$\begin{vmatrix} x, & y, & yy_1 \\ 1, & y_1+xy_2, & y_1^2+yy_2 \\ 0, & 3y_2+xy_3, & 3y_1y_2+yy_3 \end{vmatrix} = 0,$$

i.e., $$x\,(2yy_1y_3+3xy_1y_2^2-xy_1^2y_3-3yy_2^2) = y^2y_3.$$

which becomes $3xy_2^2-xy_3y_1+yy_3=0$ on dividing by the factor xy_1-y.

13700. (D. Biddle.)—Construct the triangle, having given (1) the radii of the three circles inscribed to the corner-spaces left by the incircle, (2) the perpendicular distances from the mid-points of the sides to the circumcircle.

Solution by H. W. Curjel, M.A.; *and* Professor Sanjána, M.A.

Let α, β, γ be (1) the given radii of the three circles, (2) the given perpendiculars. Then, expressing $\sin\frac{1}{2}A = \cos(\frac{1}{2}B+\frac{1}{2}C)$, (1) in terms of r, α, β, γ; (2) in terms of R, α, β, γ, we get

(1) $r = \sqrt{(\alpha\beta)}+\sqrt{(\beta\gamma)}+\sqrt{(\gamma\alpha)}$, (2) $R = \dfrac{2\alpha\beta\gamma}{2\alpha\beta+2\beta\gamma+2\gamma\alpha-\alpha^2-\beta^2-\gamma^2}$.

Hence the constructions are obvious.

[If, in (1), the radius of the incircle $= q$, the radii of the three smaller circles represent linearly squares of the roots of a cubic equation of the third term of which q is the coefficient.]

13674. (Rev. J. Cullen, B.A.)—If

$$(2p/n)^{n/(n-2)}-p\,(2p/n)^{2/(n-2)}+r = 0,$$

then $$n^n r^{n-2}-4p^n\,(n-2)^{n-2} = 0.$$

Solution by E. P. Barrett, B.A.; *and* Lionel E. Reay, B.A.

$$\begin{aligned} r &= p\,(2p/n)^{2/(n-2)}-(2p/n)^{n/(n-2)} \\ &= (p/n)^{n/(n-2)}\left\{p\,4^{1/(n-2)}\,(p/n)^{(2-n)/(n-2)}-2^{n/(n-2)}\right\} \\ &= (p/n)^{n/(n-2)}\left\{4^{1/(n-2)}\,n-2\,.\,4^{1/(n-2)}\right\}; \end{aligned}$$

therefore $$n^n r^{n-2}-4p^n\,(n-2)^{n-2} = 0.$$

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13590. (J. Woodcote West.)—(1) 2^k cards, when shuffled one by one x^x times, return to their original order x^{x-1} times: determine x. (2) If the n^{th} card from the top is the "akinete" of a pack shuffled one by one, it becomes the $(n.2^{[1-(-1)^{n/2}]/2})^{\text{th}}$ card after the pack has been shuffled once by two's.

Solution by the Proposer.

(1) Shuffled x^x, return x^{x-1}; shuffled x^{x-a}, return x^{x-a-1}; shuffled x, return 1; therefore $x = k+1$.

(2) If N is the greatest even number of "p groups" of cards contained in a pack, the position of the akinetic group (when it exists) is $(N+2)/3$, the pack being shuffled group by group, and by substituting the nearest even number contained in N/q for N in the equation

$$\left[N + n(-1)^{n+1} + 2^{[1+(-1)^n]/2}\right]/2 = n_1,$$

and expressing N in terms of the akinetic group, &c., and making p equal 1, and q equal 2, we obtain the above equation.

13570. (Rev. T. Mitchenson, B.A.)—If l, m, n be the perpendiculars from the angular points upon the respective sides a, b, c of a triangle, then $(l^3+m^3+n^3)/(a^{-3}+b^{-3}+c^{-3}) = lmn/a^{-1}b^{-1}c^{-1}$.

Solution by F. H. Peachell, B.A.; W. E. Jeffares, B.A.; *and many others.*

Now, if Δ be area of the triangle,

$$l.a = m.b = n.c = 2\Delta \quad \ldots\ldots (1).$$

$$\therefore \quad lmn\,abc = 8\Delta^3 \quad \ldots\ldots (\alpha).$$

Also $l^3 = 8\Delta^3/a^3$, &c., ... from (1).

$$\therefore \quad l^3+m^3+n^3 = 8\Delta^3(a^{-3}+b^{-3}+c^{-3}) \quad \ldots\ldots (\beta).$$

Therefore, from (α) and (β), $(l^3+m^3+n)/(a^{-3}+b^{-3}+c^{-3}) = lmn/a^{-1}b^{-1}c^{-1}$.

13730. (R. Tucker, M.A.)—PQR is a circle through three co-normal points on a parabola, and $p'q'r'$ is the "N.P." circle of the tangential triangle (pqr). It is known that both these circles pass through the vertex A. Prove that the Simson-lines of A, for the triangles PQR, $p'q'r'$, pass through fixed points. Find the locus of the intersection of these lines, when (i.) P is fixed, (ii.) the rectangle under the ordinates of Q, R is constant. (*Cf.* "Some Properties of Co-normal Points on a Parabola," *Proceedings of London Mathematical Society*, Vol. xxi, pp. 442–451.)

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Solution by the PROPOSER.

Let P be $(am_1^2, 2am_1)$; then p' is $(-am_1^2/2, am_1/2)$. The equation to PQ is $m_2y + 2x = -2am_1m_2$, and the perpendicular from A is $2y - m_2x = 0$; hence their point of intersection is

$$(m_3^2+4)\,y_3 = -2a\mu, \quad (m_3^2+4)\,x_3 = -4am_1m_2 \quad (\mu \equiv m_1m_2m_3).$$

The equation to the SIMSON-line is

$$y\,(S+8) + \mu\,(x+4a) = 0 \quad (S \equiv m_1^2 + m_2^2 + m_3^2) \ldots\ldots\ldots\ldots \text{(i.)}.$$

This passes through $(-4a, 0)$.

The equation to $p'q'$ is $m_2y - x = -am_1m_2/2$, and the perpendicular on this is $m_2x + y = 0$. Hence equation to the SIMSON-line is

$$\mu x - (S+2)\,y = a\mu \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii.)},$$

a line through the focus.

Write (i.) and (ii.) in the form

$$8y + \mu\,(x+4a) = -8y, \quad -8y + \mu\,(x-a) = 2y,$$

whence

$$S\,(2x+3a) = -10x, \quad \mu\,(2x+3a) = -6y.$$

If

$$m_2m_3 = \lambda, \quad \text{then} \quad S/2 = m_1^2 - \lambda.$$

If P is fixed, *i.e.*, m_1 constant, then the locus of the intersection of (i.) and (ii.) is the straight line

$$(2m_1^3 + 5m_1)\,x + 6y + 3am_1^3 = 0;$$

and, if λ is constant, *i.e.*, m_1 variable, then the locus is the conic

$$36y^2 + 2\lambda^2\,(5-2\lambda)\,x^2 + 3a\lambda^2\,(5-4\lambda)\,x - 9a^2\lambda^3 = 0,$$

which is an ellipse, a parabola, or a hyperbola, according as $2\lambda \lessgtr 5$.

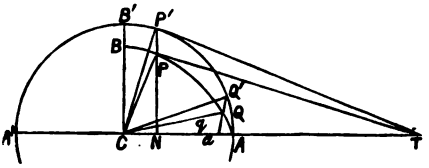
13476. (R. F. DAVIS, M.A. Suggested by Quest. 13405.)—Give the geometrical construction for an ellipse similar to, concentric with, and touching at a given point P, a given ellipse. Also show that the linear dimensions of the two curves are as CP : CD.

Solution by L. E. REAY, B.A.

Draw the auxiliary circle. Let P′ correspond to P. Then

$$\tan \text{CPT} = \tan(\text{CPN} + \text{NPT})$$

$$= \left(\frac{\text{CN}}{\text{PN}} + \frac{\text{NT}}{\text{PN}}\right) \Big/ \left(1 - \frac{\text{CN.NT}}{\text{PN}^2}\right) = \left(\frac{a}{b}\,\frac{\text{CT}}{\text{P'N}}\right) \Big/ \left(1 - \frac{a^2}{b^2}\right);$$



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therefore $$\tan CPT = \frac{ab}{(b^2-a^2)} \cdot \frac{CT}{P'N}.$$

And $$\frac{CT}{P'N} = \frac{CT}{CP'}\frac{CP'}{P'N} = \frac{1}{\cos P'CA \sin P'CA} = \frac{1}{\cos Q'CA \sin Q'CA}$$

(where $Q'CA = P'CB$); therefore, if Q corresponds to Q', we have $\angle CQt = CPT$.

Find point a on CA, such that $Ca : CA :: CP : CQ$.

Draw the ellipse with vertex a, concentric with and similar to ellipse AB. If q corresponds to Q, then $Cq = CP$; therefore, if this new ellipse is revolved through an angle QCP, q will coincide with P, and tangent at q with tangent at P. Hence this ellipse is equal and similar to the required ellipse.

Hence the construction is—Find the point P' on the auxiliary circle corresponding to P. Draw CQ' such that $Q'CA = P'CB$. Find Q corresponding to Q'.

Draw the ellipse, similar to, similarly placed, and concentric with the given ellipse, such that its radii are in the ratio CP : CQ to those of the given ellipse.

Draw a line through C making an angle with $CA = \angle PCQ$.

On this as major axis construct an ellipse equal and similar to the one just found. This is the ellipse required.

If the radius CD is drawn such that $DCA' = QCA$, then D'CP' is a right angle, and CD is conjugate to CP and equal to CQ.

Therefore the ellipses are in the ratio CP : CQ, that is, CP : CD.

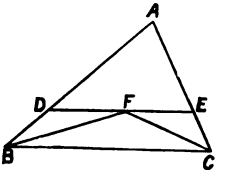
9509. (Professor Hanumanta Rau, M.A.)—Find two points D, E in the sides AB, AC of a triangle ABC, such that $BD + CE = DE$.

Solution by Professor G. B. M. Zerr.

Bisect the angles B, C, and through the intersection F of the bisectors BF, CF draw DE parallel to BC; then D, E are the points required. For the triangles BDF, CEF are isosceles; therefore

$$BD = DF,$$
$$CE = EF;$$

therefore $$BD + CE = DE.$$



13593. (W. E. Jeffares, B.A.)—Let $2c$ be the base of a spherical triangle, and β be the arc of a great circle connecting the vertex with the middle point of the base; also let θ be the angle made by β with the base.

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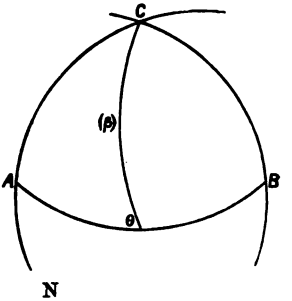
Prove that $$\frac{(1-\tan^2\frac{1}{2}c\tan^2\frac{1}{2}\beta)\tan\frac{1}{2}E}{2\tan\frac{1}{2}c\tan\frac{1}{2}\beta}=\sin\theta,$$

where E is the spherical excess.

Solution by the PROPOSER.

Let $2a$, $2b$, $2c$ be the sides of the spherical triangle ABC;



$$\frac{\sin\theta}{\sin A}=\frac{\sin 2b}{\sin\beta};$$

$$\therefore\ \sin\theta=\frac{\sin 2b}{\sin\beta}\sin A,$$

$$\sin\theta=\frac{\sin 2b}{\sin\beta}\cdot\frac{2N}{\sin 2b\sin 2c},$$

where $N=[\sin S\sin(S-2a)\times\sin(S-2b)\sin(S-2c)]^{\frac{1}{2}}$,

$$\sin\theta=\frac{2N}{\sin\beta\sin 2c}=\frac{N}{\sin\beta\sin c\cos c}.$$

Now $$\cot\tfrac{1}{2}E=\frac{1+\cos 2a+\cos 2b+\cos 2c}{2N}$$

(see TODHUNTER's *Spherical Trigonometry*, §§ 101 and 103, pp. 70 and 71, third edition).

$$N=\frac{1+\cos 2a+\cos 2b+\cos 2c}{2\cot\frac{1}{2}E};\quad\therefore\ \sin\theta=\frac{1+\cos 2a+\cos 2b+\cos 2c}{2\sin\beta\sin c\cos c\cot\frac{1}{2}E}.$$

Again, as $$\cos 2a+\cos 2b=2\cos\beta\cos c,$$

therefore $$\sin\theta=\frac{2\cos^2 c+2\cos\beta\cos c}{2\sin\beta\sin c\cos c\cot\frac{1}{2}E}=\frac{\cos c+\cos\beta}{\sin\beta\sin c\cot\frac{1}{2}E}$$

$$=\frac{(1+\cos c)(1+\cos\beta)-(1-\cos\beta)(1-\cos c)}{8\sin\frac{1}{2}\beta\cos\frac{1}{2}\beta\sin\frac{1}{2}c\cos\frac{1}{2}c\cot\frac{1}{2}E}$$

$$=\frac{\cos^2\frac{1}{2}c\cos^2\frac{1}{2}\beta-\sin^2\frac{1}{2}\beta\sin^2\frac{1}{2}c}{2\sin\frac{1}{2}\beta\cos\frac{1}{2}\beta\sin\frac{1}{2}c\cos\frac{1}{2}c\cot\frac{1}{2}E}$$

$$=\frac{1-\tan^2\frac{1}{2}\beta\tan^2\frac{1}{2}c}{2\tan\frac{1}{2}\beta\tan\frac{1}{2}c\cot\frac{1}{2}E};$$

and therefore $$\sin\theta=\frac{(1-\tan^2\frac{1}{2}\beta\tan^2\frac{1}{2}c)\tan\frac{1}{2}E}{2\tan\frac{1}{2}\beta\tan\frac{1}{2}c}.$$

13438. (Professor BARISIEN.)—Dans l'hyperbole équilatère deux cordes focales perpendiculaires ont des longueurs égales.

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Solution by L. E. Reay, B.A.; Professor A. Droz-Farny; *and others.*

Let PSP′, QSQ′ be any two focal chords of the hyperbola.

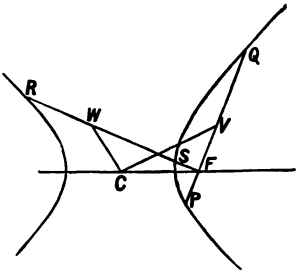
We have $$\frac{1}{SP}+\frac{1}{SP'}=\frac{1}{SQ}+\frac{1}{SQ'}.$$

$$\therefore\ \frac{SP+SP'}{SP.SP'}=\frac{SQ+SQ'}{SQ.SQ'}.$$

$\therefore$ the focal chords are in the ratio $\dfrac{SP.SP'}{SQ.SQ'}$.

But this is the ratio of the diameters parallel to the chords.

But in an equilateral hyperbola perpendicular diameters are equal; therefore the focal chords are equal.



[Mr. Curjel solves the question as follows:—Let V and W be the middle points of the perpendicular chords PQ, RS through the focus F. Join V, W to the centre C. Then CV, CW are at right angles and angles WCF, VFC are equal. Therefore W and V are equidistant from the directrix which bisects CF; but SR and PQ pass through the focus.

$\therefore$ SR = PQ.]

13707. (R. F. Davis, M.A.)—Resolve 382^4+1 into factors.

Solutions (1) *by* Lt.-Col. Allan Cunningham, R.E.; (2) *by* Joseph Blater.

Euler has shown that every *prime* factor (p) of $N=a^2+1$ is of form $p=8w+1$. Reuschle's *Tafeln complexer Primzahlen*, p. 441, shows that 17 is a divisor, and that there is no other divisor <1000. Now write

$$N'=N\div 17=1252,577,281.$$

Next form a table of numbers x of the four types

$$x=1000n+409,\ 841,\ 4159,\ 4591$$

(whose squares all end in 7281); form the differences ($N'\sim x^2$), and factorize these differences. Two of these results are

$$N'=25841^2+2\,.\,17100^2=14591^2+2\,.\,22800^2$$

$$=c_1^2+2\,.\,D_1^2=c_2^2+2D_2^2\ \text{(suppose)}\ \ldots\ldots\ldots\ldots(1).$$

Also let $N'=N_1'\,.\,N_2'$; then these are also of forms

$$N_1'=c_1^2+2d_1^2,\quad N_2'=c_2^2+2d_2^2.$$

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Multiplying N_1', N_2' together, and identifying with (1) above, there may be derived $c_2 : d_2 = C_1 \mp C_2 : D_2 \mp D_1$. With the upper signs,

$$c_2 : d_2 = 75 : 38; \quad \therefore \quad N_2 = 75^2 + 2 \cdot 38^2 = 8513;$$

whence $N_1 = 147137$ and $N = 17 \cdot 8513 \cdot 147137$.

[Mr. Joseph Blater solves the Question as follows:—
$382^4 + 1 = 21293813777$. Using Blater's Table of quarter-squares, the solution can be effected as follows:—

21294105625 is the next greatest square with argument 291850,
21293813777

291848 remainder, which is not a tabular number; ∴ we add
145925 half arg. 291851, where the fraction ½ must be neglected.
The sum 437773 is again not a tabular number; ∴ we add further
145926 half arg. 291852.
583699
145926 = half 291853, continuing to add the halves of the next
729625 arg. till finally the sum gives a square.
145927
875552
1021479
145928
1167407
1313335
145929
1459264 which is at last a tabular number with arg. 2416.

The initial arg. is 291858 arg. for sum of factors.
The final arg. is 2416 arg. for difference of factors.
The factors consist of the halves of { 294974 = 147137 = half sum of the arguments, 289442 = 144721 = ,, difference of the arguments.

144721 = 17 × 8513, 147137 is prime; therefore
$382^4 + 1 = 17 \times 8513 \times 147137$.]

13727. (W. J. C. Miller, B.A.)—Find *all* the values of x which will satisfy the equation

$$\cos^{-1}(3/x) + \cos^{-1}(4/x) + \cos^{-1}(5/x) = \pi.$$

Solution by Professor Umes Chandra Ghosh, M.A.; R. Chartres; *and* Rev. T. Mitchesson, B.A.

Let $\cos^{-1}(3/x) = \alpha$, $\cos^{-1}(4/x) = \beta$, and $\cos^{-1}(5/x) = \gamma$; then, since

$$\cos\gamma = -\cos(\alpha+\beta), \quad x(x^3 - 50x - 120) = 0.$$

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Solving the cubic equation by CARDAN's method, we find that

$$x = \sqrt[3]{p} + \sqrt[3]{q}, \quad \text{where} \quad p = \tfrac{1}{2}\left\{120 + \tfrac{20}{3}\sqrt{(-\tfrac{275}{3})}\right\},$$

and

$$q = \tfrac{1}{2}\left\{120 - \tfrac{20}{3}\sqrt{(-\tfrac{275}{3})}\right\}.$$

Now, employing the method given in page 108 of BURNSIDE and PANTON's *Theory of Equations*, of obtaining numerical values of the roots of a cubic, we see that

$$\kappa = \tfrac{20}{3}\sqrt{(\tfrac{275}{3})}, \quad \text{and} \quad \tan\phi = \tfrac{1}{15}\sqrt{(\tfrac{275}{3})};$$

$$\therefore \cos\phi = (9\sqrt{6})/25; \quad \therefore \cos\phi = 9{\cdot}9453782; \quad \therefore \phi = 28^\circ\,8'\,16'';$$

therefore

$$\tfrac{1}{3}\phi = 9^\circ\,22'\,45''.$$

Now, since the roots are $2(-\mathrm{H})^{\frac{1}{2}}\cos\frac{1}{3}\phi$, $-2(-\mathrm{H})^{\frac{1}{2}}\cos\frac{1}{3}(\pi \pm \phi)$,

by the help of the Mathematical Tables, we find that their values are $-5{\cdot}1804$, $-2{\cdot}8756$, and $8{\cdot}0558$. Since $3/x$, $4/x$, and $5/x$ are cosines of angles, the second root is inadmissible. Therefore the only admissible roots are $-5{\cdot}1804\ldots$ and $+8{\cdot}0558\ldots$.

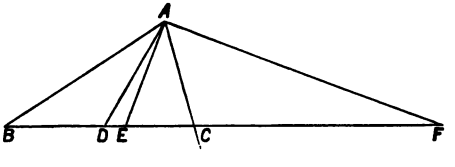
13709. (G. W. PRESTON, B.A.)—Construct a triangle, having given the base, the difference of the base-angles, and the sum of squares on the sides.

Solution by the PROPOSER, *and* L. E. REAY, B.A.

Suppose ABC the required triangle. Then, because

$$AB^2 + AC^2 = 2BD^2 + 2DA^2 = \text{a known quantity},$$

therefore AD is known.



Again, if AE, AF be the internal and external bisectors of angle A, AB, AE, AC, AF form a harmonic pencil; therefore (BECF) is a harmonic range, and D is the mid-point of BC; therefore

$$DE \cdot DF = DC^2 = \text{a constant}.$$

Also $\angle AEB = C + \frac{1}{2}A = C + \frac{1}{2}(180 - B - C) = 90^\circ + \frac{1}{2}(C - B) = \text{known } \angle$,

and

$$\angle AFB = AEB - 90^\circ = \tfrac{1}{2}(C - B) = \text{known } \angle.$$

Hence the following construction:—Take AD = the known line; on it draw two segments of circles containing respectively angles $= \frac{1}{2}(C - B)$

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and $90° + \frac{1}{2}(C - B)$. From point of intersection D, draw a line DEF cutting circles in E, F, so that $DE \cdot DF = DC^2$ (because DC = half the base). Mark off DC and produce CD to B, making DB = BC. Join AB, AC, and ABC will be required triangle.

12556. (H. J. Woodall, A.R.C.S.)—Solve the following equations:—

(1) $x^6 - 6x^5 + 25x^4 - 58x^3 + 102x^2 - 96x + 72 = 0$,

(2) $x^6 - 10x^5 + 64x^4 - 231x^3 + 616x^2 - 1000x + 1120 = 0$,

(3) $x^8 - 10x^7 + 57x^6 - 208x^5 + 552x^4 - 1048x^3 + 1472x^2 - 1344x + 768 = 0$,

(4) $x^{10} - 40x^9 + 701x^8 - 7072x^7 + 45247x^6 - 189928x^5 + 520731x^4 - 895920x^3 + 886712x^2 - 427968x + 74880 =$

(5) $x^{12} - 35^{11} + 564x^{10} - 5594x^9 + 37990x^8 - 185490x^7 + 663479x^6 - 1732915x^5 + 3217626x^4 - 4065166x^3 + 3400100x^2 - 1767360x + 436800 = 0.$

Solution by Professor S. Sircom, M.A.

The factors are as follows:—

(1) $(x^2 - x + 2)(x^2 - 2x + 6)(x^2 - 3x + 6)$,

(2) $(x^2 - x + 7)(x^2 - 4x + 8)(x^2 - 5x + 20)$,

(3) $(x^2 - x + 4)(x^2 - 2x + 4)(x^2 - 3x + 6)(x^2 - 4x + 8)$,

(4) $(x^2 - 8x + 3)(x^2 - 8x + 5)(x^2 - 8x + 13)(x^2 - 8x + 16)(x^2 - 8x + 24)$,

(5) $(x^2 - x + 1)(x^2 - 3x + 15)(x^2 - 5x + 4)(x^2 - 7x + 28)(x^2 - 9x + 26)$
$(x^2 - 10x + 10)$.

For trial purposes it is convenient to work only with the digits in the units place throughout; only those factors need be further considered which will then satisfy: nearly all those factors which are correct when the last two digits are taken will be found to be factors of the original expression. [For another Solution of this Quest., see Vol. lxiii., p. 41.]

13675. (J. Brill, M.A.)—Given the eight differential equations

$$\left.\begin{aligned} du_1 &= w_1 d\phi_1 + v_1 d\psi_1 \\ dv_1 &= u_1 d\phi_1 + w_1 d\psi_1 \\ dw_1 &= v_1 d\phi_1 + u_1 d\psi_1 \end{aligned}\right\}, \qquad \left.\begin{aligned} du_2 &= w_2 d\phi_2 + v_2 d\psi_2 \\ dv_2 &= u_2 d\phi_2 + w_2 d\psi_2 \\ dw_2 &= v_2 d\phi_2 + u_2 d\psi_2 \end{aligned}\right\},$$

$$d\phi_1 + d\phi_2 = 0, \quad d\psi_1 + d\psi_2 = 0,$$

prove that

$$v_1 w_2 + v_2 w_1 + u_1 u_2 = \text{const.},$$

$$w_1 u_2 + w_2 u_1 + v_1 v_2 = \text{const.}, \quad u_1 v_2 + u_2 v_1 + w_1 w_2 = \text{const.}$$

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Solution by Professor UMES CHANDRA GHOSH, *and many others.*

$$v_1 dw_2 + w_2 dv_1 + v_2 dw_1 + w_1 dv_2 + u_1 du_2 + u_2 du_1$$
$$= v_1(-v_2 d\phi_1 - u_2 d\psi_1) + w_2(u_1 d\phi_1 + w_1 d\psi_1)$$
$$+ v_2(v_1 d\phi_1 + u_1 d\psi_1) + w_1(-u_2 d\phi_1 - w_2 d\psi_1)$$
$$+ u_1(-w_2 d\phi_1 - v_2 d\psi_1) + u_2(w_1 d\phi_1 + v_1 d\psi_1) = 0;$$

therefore $$d(v_1 w_2) + d(v_2 w_1) + d(u_1 u_2) = 0;$$

therefore $$v_1 w_2 + v_2 w_1 + u_1 u_2 = \text{const.}$$

Similarly the other two relations can be obtained.

13694. (Professor ROGEL, LL.D.)—Prove the convergence of the series $$R \equiv \sum_{z=2,3,\ldots}^{\infty} \frac{(lz)^p}{z^r} \quad (\infty > p > 0,\ r > 1).$$

Solution by the PROPOSER.

If $$R \equiv \sum_{z=2,3,\ldots}^{\infty} \frac{(\log z)^p}{z^r} \quad (p < \infty) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

we have, in consequence of (1) or $\log z < \sqrt[n]{(n!\, z)}$, the inequation

$$R < (n!)^{p/n} \sum \frac{1}{z^{r-p/n}},$$

where $r - p/n > 1$; if $r > 1$, in that case the series R is convergent.

[Mr. CURJEL solves the Question as follows:—Applying CAUCHY's condensation test, the given series is convergent, or divergent with

$$\sum \frac{e^z z^p}{e^{rz}}, \quad i.e., \quad \sum \frac{z^p}{e^{(r-1)z}},$$

which is evidently convergent.]

13722. (Professor E. LEMOINE.)—Démontrer que dans un triangle dont les trois côtés sont a, b, c, les rayons des cercles tangents aux trois côtés r, r_a, r_b, r_c, le rayon du cercle circonscrit R, on a

$$\tfrac{1}{16}(b^2-c^2)^2(c^2-a^2)^2(a^2-b^2)^2$$
$$= \left[r^2(2R+r) + r_a r_b r_c\right]\left[r_a^2(2R-r_a) - r r_b r_c\right]\left[r_b^2(2R-r_b) - r r_c r_a\right]$$
$$\left[r_c^2(2R-r_c) - r r_a r_b\right].$$

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Solution by the PROPOSER, *and* Professor SANJÁNA, M.A.

On démontre sans difficulté la formule

$$r^2(2R+r)+r_a r_b c_c=(b+c)(c+a)(a+b)\,r/2p \quad \ldots\ldots\ldots\ldots(1);$$

par transformation continue de (1) en A en B et en C on a respectivement

$$-r_a^2(2R-r_a)+rr_b r_c=-(b+c)(c-a)(a-b)\,r_a/2(p-a)\ldots\ldots(2),$$

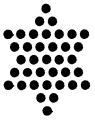
$$-r_b^2(2R-r_b)+rr_a r_c=-(b-c)(c+a)(a-b)\,r_b/2(p-b)\ldots\ldots(3),$$

$$-r_c^2(2R-r_c)+rr_a r_b=-(b-c)(c-a)(a+b)\,r_c/2(p-c)\ldots\ldots(4).$$

Multipliant membre à membre (1), (2), (3), (4) après avoir changé le signe des deux membres de (2), (3), (4), et remarquant que

$$rr_a r_b r_c=p(p-a)(p-b)(p-c),$$

on a la formule cherchée.

13744. (J. WOODCOTE WEST.) — Express the nth number of the form which is also of the form  .

Solution by the PROPOSER.

It depends on making $2\square+1=\square$, and the nth term is

$$\left[\left\{(3+2\sqrt{2})^{4n-3}-(3-2\sqrt{2})^{4n-3}\right\}^2-8\right]/4! \quad \text{or} \quad (4N-1)/3,$$

where N is the $(4m-3)$-th square of the form $a(a+1)/2$. The first four numbers are

1,
1,884,961,
2,510,229,508,033,
3,342,907,779,059,328,481.

13745. (Rev. T. MITCHESON, B.A.)—In regard to the triangle ABC, whose sides are a, b, c, a point P moves so that $PB^2=PA^2+PC^2$. Determine the locus of P, and the condition of its being a real curve.

Solutions (1) *by* Professor SANJÁNA, M.A.; (2) *by* LIONEL E. REAY, B.A.

(1) Take BA, BC as axes of coordinates; then the locus is

$$x^2+y^2+2xy\cos B=(x-a)^2+y^2+2(x-a)\,y\cos B$$
$$+x^2+(y-c)^2+2x(y-c)\cos B,$$

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i.e., $(x-a)^2+(y-c)^2+2(x-a)(y-c)\cos B-2ac\cos B=0$.

Thus in general the locus is a circle, centre (a, c), radius $(2ac\cos B)^{\frac{1}{2}}$. This reduces to a point when $B = 90°$, and becomes impossible when $B > 90°$.

(2) If coordinates of P referred to the given triangle are (x, y, z), then

$$-PA^2 = a^2yz+b^2z(x-1)+c^2(x-1)y = a^2yz+b^2zx+c^2xy-(b^2z+c^2y).$$

Similarly for PB^2 and PC^2.

Hence, substituting these values, we get locus of P to be

$$a^2yz+b^2zx+c^2xy-x(b^2-c^2)-y(a^2+c^2)-z(b^2-a^2)=0.$$

This is a circle; it is real if the coefficients of x, y, and z are negative. This is the case when $b^2 > c^2$ and $> a^2$; therefore the locus is real when b is the greatest side of the triangle.

13609. (Professor Sanjána.)—Prove that

$$\int_0^\infty \sin(x^n)\,dx = b\,\Gamma(1+1/n),$$

where b is the real coefficient of the imaginary part of $(-1)^{\frac{1}{2n}}$; and hence find the value of the integral to four places of decimals when n is 2 or 3.

Solution by Rev. J. Cullen, B.A.; *and* H. W. Curjel, M.A.

$$I = 1/n\int_0^\infty \sin u\, u^{1/n-1}\,du, \quad \text{where } u = x_n.$$

Therefore $I = 1/n\,\Gamma(1/n)\sin\pi/2n = b\,\Gamma(1+1/n)$,

since $b = \sin\pi/2n$.

Putting $n = 2$, $I = \frac{1}{2}(\frac{1}{2}\pi)^{\frac{1}{2}} = \cdot6266$,

$n = 3$, $I = \frac{1}{3}\Gamma(1\cdot333) = \cdot4465$.

[See Williamson's *Int. Calc.*, Art. 124, and the table for $\log\Gamma(p)$, Art. 125.]

13610. (W. J. C. Miller, B.A.)—A square is inscribed in a triangle, a second square in the triangle whose base is the side of the first square and vertex that of the given triangle, and so on *ad inf.* Find the sum of the areas of all these squares.

Solution by J. Woodcote West; *and* H. W. Curjel, M.A.

When the squares stand one upon another, the area of the nth square is $b^2[h/(h+b)]^{2n}$, where b is the base of the triangle, and h its height; and the sum of all these areas *ad infinitum* is $(b \,.\, h^2)/(2h+b)$.

[See Townsend, p. 114.]

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13558. (R. F. Davis, M.A.)—Resolve $(x^2+3)^4+(4x)^4$ into biquadratic factors.

Solution by Rev. J. Cullen, B.A.; *and* F. H. Peachell, B.A.

The expression $= \{(x^2+3)^2+i\,(4x)^2\}\{(x^2+3)^2-i\,(4x)^2\}$.

Hence the quadratic factors are $x^2+3\pm(\pm i)^{\frac{1}{2}}4x$, and, since

$$(\cos\theta\pm i\sin\theta)^{\frac{1}{2}} = \cos\tfrac{1}{2}\theta\pm i\sin\tfrac{1}{2}\theta,$$

putting $\theta=\frac{1}{2}\pi$, we have $(\pm i)^{\frac{1}{2}} = 2^{-\frac{1}{2}}(1\pm i)$. Therefore the biquadratic factors are $(x^2+3\pm 2^{\frac{1}{2}}x)^2+8x^2$.

[The Proposer's result is as follows:—The given expression

$$= x^8+12x^6+310x^4+108x^2+81;$$

and is the product of two factors $x^4\pm 4x^3+14x^2\mp 12x+9$.]

6618. (J. W. Russell, M.A.)—Show that the six anharmonic ratios of the four lines $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=0$ are the roots of the equation in z,

$$(1/\theta^2)-(1/\theta^3) = \tfrac{4}{27}-(4T^2/S^3),$$

where $\theta = z+(1/z)-1$ and S and T are the invariants of the given quantic.

Solution by H. J. Woodall, A.R.C.S.

Let α, β, γ, δ be the roots of the given biquadratic. Then the anharmonic ratios will be

$$z_1 = (\alpha-\beta)(\gamma-\delta)/(\alpha-\delta)(\gamma-\beta) = -(m-n)/(l-m), \text{ say},$$

$$z_2 = -(n-l)/(m-n),\quad z_3 = -(l-m)/(n-l),\quad z_4 = -(l-m)/(m-n),$$

$$z_5 = -(m-n)/(n-l), \text{ and } z_6 = -(n-l)/(l-m);$$

therefore $z_1z_2z_3 = -1 = z_4z_5z_6$, and $z_4 = 1/z_1$, $z_5 = 1/z_2$, $z_6 = 1/z_3$.

$$\text{Sum of the } z\text{'s} = -\left(\frac{m-n}{l-m}+\frac{n-l}{m-n}+\frac{l-m}{n-l}+\frac{l-m}{m-n}+\frac{m-n}{n-l}+\frac{n-l}{l-m}\right) = 3.$$

Sum of the products (2 at a time) of the z is equal to $-3-S_1S_2$; so the other sums of the products, 3 . 4 . 5 . 6 at a time, are $-11-2S_1S_2$, $-3-S_1S_2$, 3, and 1, where

$$S_1 = z_1+z_2+z_3,\quad S_2 = z_2z_3+z_3z_1+z_1z_2.$$

The equation in z is therefore a reciprocal equation, and can be at once written down thus:

$$(z^3+z^{-3})-3(z^2+z^{-2})+(z+z^{-1})(-3-S_1S_2)+(11+2S_1S_2) = 0;$$

if we put $\theta = z+z^{-1}-1$, we get

$$\theta^{-2}-\theta^{-3} = 1/(9+S_1S_2) = \tfrac{4}{27}-4T^2/S^3,$$

as required.

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It may be useful to quote some formulæ:

$$(l-m)^2(m-n)^2(n-l)^2 = \Pi(\alpha-\beta)^2 = 256(S^3-27T^2)/a^6;$$

$$(l-m)^2+(m-n)^2+(n-l)^2 = 24S/a^2;$$

$$S = ae-4bd+3c^2; \quad T = ace+2bcd-ad^2-b^2e-c^3.$$

[The PROPOSER solves the question as follows:—As each anharmonic ratio is an absolute invariant, the equation must depend entirely on the value of S^3/T^2.

Take two of the factors of the given equation as X and Y. It then becomes $4BX^3Y+6CX^2Y^2+4DXY^3 = 0$,

or $4DXY(Y-M_1X)(Y-M_2X) = 0$,

where $M_1+M_2 = -3C/2D$ and $M_1M_2 = B/D$.

One value of z is M_1/M_2;

$$\therefore \quad C/D = -\tfrac{3}{2}M_2(1+z), \quad B/D = M_2^2z.$$

Substituting these values in $S^3/T^2 = (3C^2-4BD)^3/(2BCD-C^3)^2$,

we get $S^3/T^2 = 108\theta^3/(4\theta^3-27\theta+27)$,

which reduces at once to the given equation.

Hence this ratio, and by parity of reasoning every other ratio, satisfies the given equation.]

13735. (R. F. MUIRHEAD.) — (1) The roots of the equations $x^2-5x+6 = 0$ and $x^2-6x+5 = 0$ are all positive integers. Are there any other pairs of equations of the form $x^2-px+q = 0$, $x^2-qx+p = 0$ which have the same property? If we denote by α, β; α', β' the roots of the equation, the question may be stated in the form: Find all the positive integral solutions of the two simultaneous equations $\alpha+\beta = \alpha'\beta'$; $\alpha'+\beta' = \alpha\beta$. (2) Extend the preceding to the case of pairs of equations like $x^2 \pm px \pm q = 0$, $x^2 \pm qx \pm p = 0$.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Any pair out of the *four* pairs of equations, given by the changes of sign in $x^2 \pm px \pm q = 0$, with *integer* roots (α, β),

$x^2 \pm qx \pm p = 0$, with *integer* roots (α', β'),

can coexist (1) when $\alpha = \alpha'$, and also (2) when $\alpha = -\alpha'$. The four pairs may be distinguished as (i.), (ii.), (iii.), (iv.). They involve severally

(i.) $\alpha+\beta = \alpha'\beta'$, $\alpha'+\beta' = \alpha\beta$; (ii.) $\alpha+\beta = -\alpha'\beta'$, $\alpha'+\beta' = -\alpha\beta$;

(iii.) $\alpha+\beta = -\alpha'\beta'$, $\alpha'+\beta' = \alpha\beta$; (iv.) $\alpha+\beta = \alpha'\beta'$, $\alpha'+\beta' = -\alpha'\beta'$.

But, as (iv.) is derivable from (iii.) by interchanging α, β with α', β', it need not be further considered.

Case (1). $\alpha = \alpha'$. This gives

In (i.): $\alpha = -1$, $\beta'+\beta = +1$. In (ii.): $\alpha = +1$, $\beta'+\beta = -1$.

In (iii.): $\beta'+\beta = \pm 1$, $\alpha = 1 \mp 2\beta$; or $\beta'+\beta = \pm 2$, $\alpha = 1 \mp \beta$.

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Case (2). $\alpha = -\alpha'$. This gives

In (i.): $\beta' - \beta = \pm 1$, $\alpha = -1 \mp 2\beta$; or $\beta' - \beta = \pm 2$, $\alpha = 1 \mp \beta$.

In (ii.): $\beta' - \beta = \pm 1$, $\alpha = 1 \pm 2\beta$; or $\beta' - \beta = \pm 2$, $\alpha = 1 \pm \beta$.

In (iii.): $\alpha = +1$, $\beta' = 1 + \beta$.

In all the above solutions there is some generality, as one of β, β' is left arbitrary; the four roots cannot all be of same sign. Another (particular) set of solutions is given by

	(i.)	(ii.)	(iii.)	(iv.)
(α, β)	$= (1, 5)$,	$(-1, -5)$,	$(-1, -5)$,	$(1, 5)$;
(α', β')	$= (2, 3)$,	$(-2, -3)$,	$(1, -6)$,	$(-1)\,6)$.

13741. (W. E. Jeffares, B.A.)—Through a given point, draw a transversal across the four sides and diagonals of a quadrilateral so that it may be the centre of involution of the points of intersection.

Solution by Lionel E. Reay, B.A.

The points in which the transversal cuts the conic through the given point and the four angular points of the quadrangle are a pair of points in the involution. Hence the required transversal cuts the conic in one point at infinity, and is therefore parallel to an asymptote.

Hence, if the conic through the five points is an hyperbola, there are two solutions; if it is a parabola, the two positions coincide and the required transversal is the diameter through the point. If the conic is an ellipse, there is no real solution.

The constructions for the real cases are known theorems.

13743. (R. W. D. Christie.)—Prove that

$$(\omega_4 + \omega_5)^{2n-1} - (\omega_2 + \omega_3)^{2n-1} / (\omega_4 + \omega_5) - (\omega_2 + \omega_3),$$

these being the unreal roots of $x^5 + 1 = 0$, always equals two integral squares, and find them.

Solution by Professor Sircom, M.A.

The imaginary fifth roots of -1 are given by

$$x^2 - \tfrac{1}{2}(1 + \sqrt{5})\,x + 1 = 0 \quad \text{and} \quad x^2 - \tfrac{1}{2}(1 - \sqrt{5})\,x + 1 = 0,$$

whence $\omega_4 + \omega_5 = \tfrac{1}{2}(\sqrt{5} + 1)$ and $\omega_2 + \omega_3 = -\tfrac{1}{2}(\sqrt{5} - 1)$;

$$\omega_4 + \omega_5 - \omega_2 - \omega_3 = \sqrt{5},$$

and we have $\left\{\left[\tfrac{1}{2}(\sqrt{5} + 1)\right]^{2n-1} + \left[\tfrac{1}{2}(\sqrt{5} - 1)\right]^{2n-1}\right\} / \sqrt{5}.$

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This is the coefficient of x^{2n-2} in the expansion of $1/(1-x-x^2)$, the roots of x^2-x-1 being $\frac{1}{2}(1\pm\sqrt{5})$. Then

$$1/(1-x-x^2) = 1+x+2x^2+3x^3+5x^4+8x^5+13x^6+21x^7+34x^8+\ldots(1),$$

and $\quad 2 = 1^2+1^2, \quad 5 = 2^2+1^2, \quad 13 = 3^2+2^2, \quad 34 = 5^2+3^2,$ &c., derived from the squares of the coefficients of (1).

The coefficients of even powers of x are those of the expansion of

$$(1-x)/(1-3x+x^2) = 1+2x+5x^2+13x^3+34x^4+\ldots.$$

To sum the series $1+x+2^2x^2+3^2x^3+5^2x^4+8^2x^5+\ldots$, let

$$3^2 = 2^2p+q+r, \quad 5^2 = 3^2p+2^2q+r, \quad 8^2 = 5^2p+3^2q+2^2r,$$

then $p=2$, $q=2$, $r=-1$, giving the next coefficient 13^2; then

$$\frac{1-x}{1-2x-2x^2+x^3} = 1+x+2^2x^2+3^2x^3+5^2x^4+\ldots,$$

$$\frac{(1+x)(1-x)}{1-2x-2x^2+x^3} = \frac{1-x}{1-3x+x^2} = 1+(1^2+1^2)x+(2^2+1^2)x^2+(3^2+2^2)x^3+\ldots,$$

which proves the theorem, and the given expression is equal to $a_{n-1}^2+a_{n-2}^2$ where a_n is the coefficient of x^n in (1).

9169. (The late W. J. C. Sharp, M.A. Suggested by Quest. 2941.) —If, in the ternarian partitionments of n, which must be of one of the forms *aaa*, *aab*, *abc*, 1 be put for each of the first, 3 for each of the second, and 6 for each of the third, the sum will be $\frac{1}{2}(n-1)(n-2)$. Or generally, if the coefficient of each of the forms of the partitions of n into p parts be represented by the coefficient of that form in $(a+b+\ldots \text{ to } p \text{ terms})^p$, the sum of all the partitions will be represented by the coefficient of x^n in $x^p(1-x)^{-p}$.

Solution by H. J. Woodall, A.R.C.S.

Such ternarian partitionments of n will consist of those where the first is 1, 2, ... $(n-2)$, of which there will be $(n-2)$, $(n-3)$, ..., respectively, and thus $\frac{1}{2}(n-1)(n-2)$ altogether. So, the quaternarian partitionments of n will be $= \Sigma\frac{1}{2}(n-2)(n-3) = \frac{1}{6}(n-1)(n-2)(n-3)$, and so on. The p-arian partitionments of n will be $= (n-1)(n-2)\ldots(n-p+1)/(p-1)!$ which is the coefficient of x^n in the expansion of $x^p(1-x)^{-p}$.

8476. (Professor Mahendra Nath Ray, M.A., LL.B.)—Show how to inscribe a pentagon in a given circle, such that the mid-points of its sides may lie on another circle.

Solution by H. J. Woodall, A.R.C.S.

Take three false positions which will determine on the circle two homographic systems of points, the double points of which systems will be the angular points of the required pentagon (it is thus evident that, generally, there will be two pentagons satisfying the given condition).

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The construction is facilitated by observing that, if AB, BC be two consecutive sides of one of the pentagons, then the mid-points A′, B′ will lie at the intersection of the second circle and a circle passing through B and the centre of the first given circle.

13718. (G. Hale Puckle, M.A.)—If (with the conditions of Quest. 13572, Vol. lxviii., p. 62) $\alpha x + \beta y = 0$, $\beta x - \alpha y = 0$ ..... (1, 2) are taken as the axes of x and y, the equation

$$ax^2 + 2hxy + by^2 + c = 0 \text{ ....(3)}$$

becomes $(a - a^2)x^2 + (b + a^2)y^2 + c = 0$.....(4);

and the axis of y is the axis of real foci.

Solution by the Proposer *and* Professor Sanjána, M.A.

Equations (1, 2), written as one locus, become

$$\alpha\beta(x^2 - y^2) - (\alpha^2 - \beta^2)xy = 0;$$

then, since $\alpha^2 - \beta^2 = a - b$, $\alpha\beta = h$, (1, 2) are the bisectors of the angles between the asymptotes, and are the axes of the curve, hence the term involving xy will disappear from (3). Squaring (1) and subtracting from (3), and squaring (2) and adding to (3), we obtain two circles, whose radii are the intercepts of the locus on the new axes, &c. Since the directrix is perpendicular to (2), (2) is the axis of real foci.

10932. (Professor Coupeau.)—Mener par deux points donnés deux parallèles qui interceptent sur un cercle donné une corde de longueur connue.

Solution by R. Knowles, B.A.; Professor Zerr; *and others.*

Let a, b; a', b, be the coordinates of the fixed points; $x^2 + y^2 = c^2$, the equation to the circle; $hx + ky = c^2$, $h_1x + k_1y = c^2$ the equations to the chords; since they pass through a, b; a', b, and are parallel, we have

$$h = (c^2 - bk)/a, \quad h_1 = c^2(c^2 - bk)/(a'c^2 - a'bk + abk),$$

$$k_1 = ac^2k/(a'c^2 - a'bk + abk),$$

and the equations to the chords become

$$(c^2 - bk)x + aky = ac^2, \quad (c^2 - bk)x + aky = (a'c^2 - a'bk + abk)\text{.....(1, 2)};$$

but $h^2 + k^2 = h_1^2 + k_1^2 = \text{constant}$; therefore $a^2k^2 + (c^2 - bk)^2 = \text{constant}$; and, substituting the values of k thus obtained in (1, 2), we get the required equations; evidently two lines can be drawn fulfilling the conditions.

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13729. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Factorize
$N_1 = 31,249,999,999,999$ and $N_2 = 31,999,999,999,999,999$.

Solution by the PROPOSER.

By a formula ascribed to M. AURIFEUILLE (published in ED. LUCAS's paper, *Sur la Série récurrente de Fermat*, 1879, pp. 5, 6),

$$(x^5-y^5)\div(x-y) = \mathrm{L}.\mathrm{M},$$

where
$$\mathrm{L} = (x^2+3xy+y^2)-(5xy)^{\frac{1}{2}}.(x+y),$$
$$\mathrm{M} = (x^2+3xy+y^2)+(5xy)^{\frac{1}{2}}.(x+y);$$

and these factors are *both* real when $5xy = \square$.

(1) Here $N_1 = (500^5-1^5) = (500-1).L_1M_1$,
$L_1 = (500^2+3.500+1^2)-50.(500+1)$,
$M_1 = (500^2+3.500+1^2)+50.(500+1)$;
therefore $N_1 = 499.226,451.276,551 = 499.226,451.(11.31.811)$.

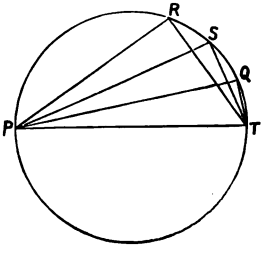
(2) Here $N_2 = (2000^5-1^5) = (2000-1).L_2M_2$,
$L_2 = (2000^2+3.2000+1^2)-100.(2000+1)$,
$M_2 = (2000^2+3.2000+1^2)+100.(2000+1)$,
therefore $N_2 = 1999.3,805,901.4,206,101$
$= 1999.(11.31.11161).4,206,101$.

13762. (Rev. T. MITCHESON, B.A.)—Let PT be a diameter of a circle, PS a chord, and PQ, PR two other chords, one on either side of PS, and making equal angles with it. Prove that

$$\left(\frac{PQ+PR}{2PS}\right)^2+\left(\frac{PQ-PR}{2ST}\right)^2 = 1.$$

Solution by W. J. CONSTABLE, M.A.; H. W. CURJEL, M.A.; *and others.*

Let $\angle SPT = \phi$,
$\angle RPS = \angle SPQ = \alpha$.
Then $RPT = \phi+\alpha$, $QPT = \phi-\alpha$.



$$\therefore \left(\frac{PQ+PR}{2PS}\right)^2+\left(\frac{PQ-PR}{2ST}\right)^2$$
$$= \left[\frac{\cos(\phi-\alpha)+\cos(\phi+\alpha)}{2\cos\phi}\right]^2 + \left[\frac{\cos(\phi-\alpha)-\cos(\phi+\alpha)}{2\sin\phi}\right]^2$$
$$= \cos^2\alpha+\sin^2\alpha = 1.$$

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[Mr. TUCKER solves the Question as follows:—
To diameter unity,

$$PQ = \cos(\phi - \alpha), \quad PR = \cos(\phi + \alpha), \quad PS = \cos\phi, \quad ST = \sin\phi.$$

$$\therefore \quad (PQ + PR)/2PS = \cos\alpha, \quad (PQ - PR)/2ST = \sin\alpha.$$ Hence, &c.

13761. (R. F. MUIRHEAD.)—Prove that in any triangle

$$\sin A + \sin B + \sin C = 2(\cos\tfrac{1}{2}A + \cos\tfrac{1}{2}B + \cos\tfrac{1}{2}C)(\sin\tfrac{1}{2}A + \sin\tfrac{1}{2}B + \sin\tfrac{1}{2}C - 1)$$

and, more generally, for all values of A, B, C, D,

$$\sin A + \sin B + \sin C + \sin D = 2(\cos\tfrac{1}{2}A + \cos\tfrac{1}{2}B + \cos\tfrac{1}{2}C + \cos\tfrac{1}{2}D)(\sin\tfrac{1}{2}A + \sin\tfrac{1}{2}B + \sin\tfrac{1}{2}C + \sin\tfrac{1}{2}D) - 4\sin\tfrac{1}{4}(A + B + C + D)\left\{\cos\tfrac{1}{4}(A + B - C - D) + \cos\tfrac{1}{4}(A + C - B - D) + \cos\tfrac{1}{4}(A + D - B - C)\right\}.$$

Solution by LIONEL E. REAY, B.A.; H. J. WOODALL, A.R.C.S.; *and others.*

$$2(\cos\tfrac{1}{2}A + \cos\tfrac{1}{2}B + \cos\tfrac{1}{2}C)(\sin\tfrac{1}{2}A + \sin\tfrac{1}{2}B + \sin\tfrac{1}{2}C - 1)$$
$$= \Sigma\, 2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A + 2\Sigma(\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B + \cos\tfrac{1}{2}B\sin\tfrac{1}{2}A) - 2\Sigma\cos\tfrac{1}{2}A$$
$$= \Sigma\sin A + 2\Sigma\sin\tfrac{1}{2}(A + B) - 2\Sigma\cos\tfrac{1}{2}A = \Sigma\sin A.$$
$$2(\Sigma\cos\tfrac{1}{2}A)(\Sigma\sin\tfrac{1}{2}A) = \Sigma\sin A + 2\Sigma\left\{\cos\tfrac{1}{2}(A + B) + \cos\tfrac{1}{2}(C + D)\right\}$$
$$= \Sigma\sin A + 4\cos\tfrac{1}{4}(A + B + C + D)\,\Sigma\cos\tfrac{1}{2}(A + B - C - D).$$

13757. (G. HALE PUCKLE, M.A.)—If $\phi(xy) = 0$ represents a parabola, show that either of the equations

$$(a + b)(ax + hy + g) + G = 0, \quad (a + b)(hx + by + f) + F = 0,$$

with $$(a + b)(gx + fy + c) - (A + B)/2 = 0,$$

will determine the focus, where $A = bc - f^2$, &c.; also, that the two first equations represent the axis, the equation to which may be written

$$a(ax + hy + g) + h(hx + by + f) = 0.$$

Solution by the PROPOSER.

From the equation $(\rho^2 + a)(\rho^2 + b) = h^2$ (see Question 13535, No. 437), the appropriate value of ρ^2 is $-(a + b)$, since, when Δ is positive, a and b are negative. Hence $\alpha^2 = -b$, $\beta^2 = -a$, $\phi^2 = 0$,

and $$g\alpha + f\beta + \Delta^{\frac{1}{2}}\gamma = \phi^2\gamma = 0;$$

and the equation $$\rho^2(ax + hy + g) + a\Delta^{\frac{1}{2}} = 0$$

becomes $$(a + b)(ax + hy + g) + g\alpha^2 + f\alpha\beta = 0;$$

and, since $$g\alpha^2 + f\alpha\beta = hf - bg = G,$$

we have the first equation. The other two are obtained from the equation to the directrix of the parabola, considered as the polar of the focus.

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13771. (G. W. Preston, B.A.)—A, B are two given points, and CD is a given straight line. Show how to draw two lines AP, BQ to meet CD in P, Q, and to contain an angle equal to a given angle. (A, B are on same side of CD, and AP must equal BQ.)

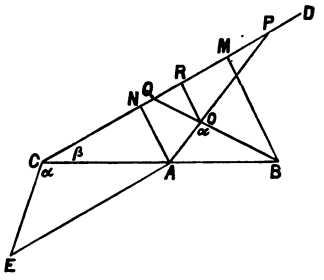
Solution by W. C. Stanham, B.A.; Professor A. Droz-Farny; *and many others.*

Draw the perpendiculars AN, BM to CD. From C (the point where CD meets AB) draw CE, making with CA the given angle α, taking E such that

$$CE : CA = AN : BM.$$

Draw AP and BQ to make the angles APC, BQD = CAE, CEA respectively. Then

QOP clearly $= \alpha$.



And, since the triangles QOP and ECA are similar,

$$OP : OQ = CA : CE = BM : AN;$$

and OR : BM = OQ : BQ (if OR is drawn perpendicular to CD); also OR : AN = OP : AP; therefore AP = BQ.

If we make ACE $= \pi - \alpha$, we obtain another solution.

13770. (E. P. Barrett, B.A.)—Let AQ be the internal bisector of the angle A in the triangle ABC, and PQ a transversal cutting AB in P and AC in R. Show that rectangle AR . BP : rect. AP . CR :: AB : AC.

Solution by H. W. Curjel, M.A.; C. Joss, M.A.; *and many others.*

$$AR . BP : AP . CR = \triangle BPR : \triangle CPR = BQ : QC = AB : AC.$$

[This is an exceedingly pretty solution.—Ed.]

The Proposer solves the Question by reference to the well-known property of a transversal,

$$AR . CQ . BP = AP . BQ . CR.$$

11822. (Professor Sircom, M.A.)—Obtain the current function for the motion of the cylinder whose right section is $(x^2+y^2)^2 = a^2(3x^2-y^2)$ along the axis of x in an infinite liquid under no forces. Also for the cylinder $(x^2+y^2)^2 = a^2(x^2-3y^2)$ moving along the axis of y.

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Solution by the PROPOSER.

The current function for two equal doublets, their axes in the same direction, at points $+a$, $-a$ along OX will be

$$\mu\left(\frac{y}{(x-a)^2+y^2}+\frac{y}{(x+a)^2+y^2}\right) \quad \ldots\ldots\ldots\ldots\ldots (1);$$

and, if fluid is moving with a velocity V along XO, or, which is the same thing, if there is a doublet at infinity in the opposite direction to the first two, the current function being $-Vy$, we shall have

$$\psi = \mu y\left(\frac{2(x^2+y^2+a^2)}{[(x-a)^2+y^2][(x+a)^2+y^2]}-\frac{V}{\mu}\right),$$

which, if $V/2\mu = 1/a^2$, becomes

$$\psi = Vy\,\frac{(x^2+y^2)^2-a^2(3x^2-y^2)}{[(x-a)^2+y^2][(x+a)^2+y^2]},$$

and ψ vanishes over the lemniscate $(x^2+y^2)^2 = a^2(3x^2-y^2)$, which is therefore a bounding curve of the series ψ = const.; hence this series gives the stream lines of fluid moving with velocity V past the curve; and the curves (1) = const. give the stream lines for the cylinder moving with velocity V through the liquid.

The portions of the curves ψ within the boundary give the stream lines of doublets at the foci of the lemniscate; and the image of each of these in the curve is a doublet at infinity. The stream lines are easily drawn by the usual process.

If we take two equal doublets at $+a$, $-a$, their axes parallel to OY, the stream function is $\mu\left(\frac{x-a}{(x-a)^2+y^2}+\frac{x+a}{(x+a)^2+y^2}\right)$, which by the same process would give $(x^2+y^2)^2 = a^2(x^2-3y^2)$ as the bounding curve; but, as this passes through the doublets, and does not include them, we do not obtain the stream function for the curve moving through the liquid, and the solution fails in this case.

The closed curves in the diagram are the stream lines when the cylinder moves through the liquid, their general equation being

$$\{(x-a)^2+y^2\}\{(x+a)^2+y^2\} = 2cay(x^2+y^2+a^2),$$

and are calculated (with the exception of the dotted curve drawn to show a double tangent) for values of the parameter c in the harmonic progression 4, 2, $\frac{4}{3}$, 1, $\frac{4}{5}$, &c.

The curves with asymptotes parallel to OX are the curves ψ = const., and are the stream lines of liquid flowing past the cylinder; they are drawn with equidistant asymptotes, their general equation being

$$\{(x-a)^2+y^2\}\{(x+a)^2+y^2\} = \frac{8ay}{8y-n}(x^2+y^2+a^2).$$

The ovals, giving the fluid motion due to doublets inside the bounding curve, belong to infinite branches on the opposite side of OX.

The points of contact of double tangents to both series of curves lie on the curve $(x^2+y^2+a^2)^2 = 4a^2(y^2+a^2)$.

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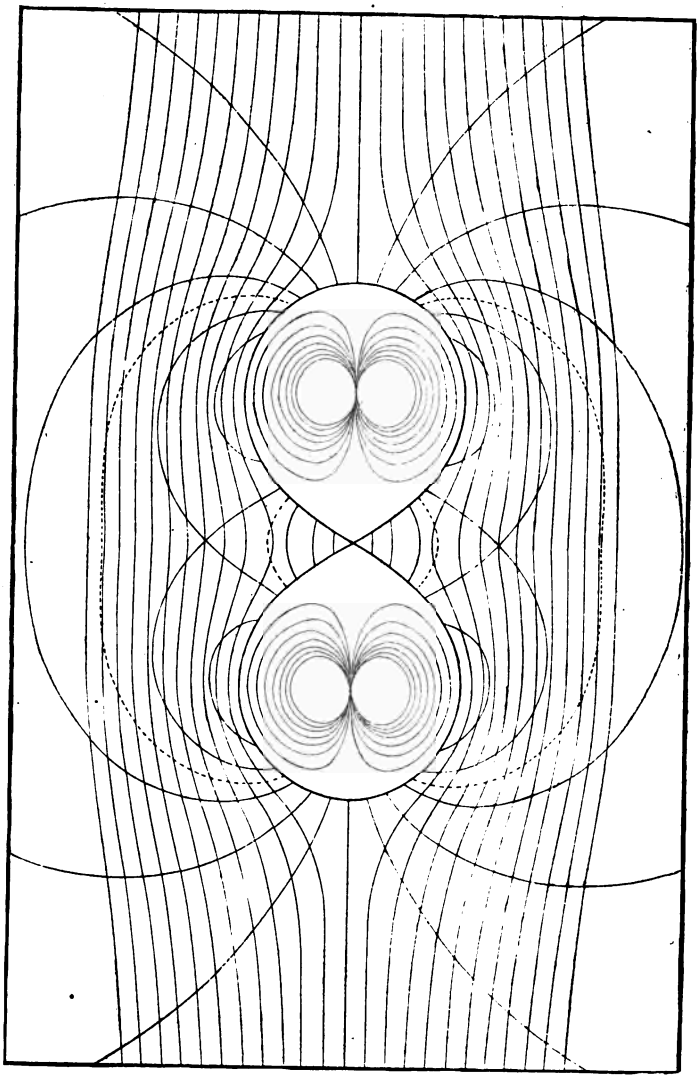

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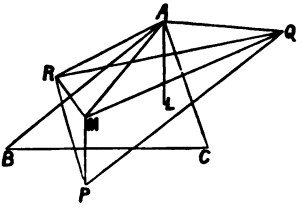
13765. (Rev. J. CULLEN, B.A.)—Adopting the notation $L = O_M$ to mean that L is the circumcentre of the triangle formed by the images of the point M, with respect to the sides of a given triangle ABC, prove that then $M = O_L$; also establish the following forms:—$\Omega = O_{\Omega'}$, $\Omega' = O_\Omega$; $P = O_O$, $O = O_P$; $G = O_K$, $K = O_G$; $I = O^I$, where Ω, P, &c., have their usual meanings with regard to ABC.

Solution by G. N. BATES, B.A.; Professor A. DROZ-FARNY; *and others.*

Let PQR be the triangle formed by the images of M.



Then $AR = AM = AQ$;

therefore perpendicular AL from A on RQ bisects it at right angles, *i.e.*, circumcentre of PQR is on AL.

Then, because A is centre of circle RMQ;

$\angle MQR = MAB$, but $MQR = LAC$;

therefore $\angle LAC = MAB$,

i.e., L is on isogonal conjugate of AM. Similarly for B and C; therefore L and M are inverse points;

therefore, if $L = O_M$, $M = O_L$;

and therefore $\Omega = O_{\Omega'}$, $\Omega' = O_\Omega$, &c.

13726. (Professor K. VENKATA RAMANAIYA.)—Find the sum of the cubes of the first n natural numbers without assuming the sum of the squares of the first n natural numbers.

Solutions (1) *by* C. Joss, M.A.; (2) *by the* PROPOSER; (3) *by* W. J. DOBBS, M.A.

(1) $\Sigma n^3 = (\Sigma n^3 + 3\Sigma n^2 + 2\Sigma n) - (3\Sigma n^2 + 3\Sigma n) + \Sigma n$

$= (1.2.3 + 2.3.4 + \ldots) - 3(1.2 + 2.3 + \ldots) + (1 + 2 + 3 + \ldots$

$= \frac{1}{4}\{n(n+1)(n+2)(n+3)\} - \frac{1}{3}\{3n(n+1)(n+2)\} + \frac{1}{2}\{n(n+1)\}$

$= \left[\frac{1}{2}\{n(n+1)\}\right]^2$.

Let $1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = S$;

then $2.3.4 + 3.4.5 \ldots = S - 1.2.3$;

therefore, subtracting

$-3\{2.3 + 3.4 + \ldots \text{ to } (n-1) \text{ terms}\} + n(n+1)(n+2) = 6$;

therefore $1.2 + 2.3 + 3.4 + \ldots$ to n terms $= \frac{1}{3}\{n(n+1)(n+2)\}$.

So likewise $1.2.3 + 2.3.4 + \ldots = \frac{1}{4}\{n(n+1)(n+2)(n+3)\}$.

So Σn^4, Σn^5, ...

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(2) Let $s = 1^2+(1+2)^2+(1+2+3)^2+\ldots\{1+2+3+\ldots(n-1)^2\}$
$+(1+2+3+\ldots n)^2$ .....(I.),
$s = 0^2+1^2+(1+2)^2+\ldots(1+2+3+\ldots n-2)^2+(1+2+3+\ldots n-1)^2$
$+(1+2+3+\ldots n)^2$.....(II.).

Subtracting (II.) from (I.) vertically, we find that the nth term of the resulting series $= (1+2+3+\ldots n)^2-(1+2+3+\ldots n-1)^2$
$= \left[\frac{1}{2}\{n(n+1)\}\right]^2-\left[\frac{1}{2}\{n(n-1)\}\right]^2 = n^3$;
therefore first term $= 1^3$, second term $= 2^3$, ... nth term $= n^3$, $(n+1)$th term is $-(1+2+3+\ldots n)^2 = -\left[\frac{1}{2}\{n(n+1)\}\right]^2$.
The left-hand member is $s-s = 0$; therefore
$0 = 1^3+2^3+3^3+\ldots n^3-\left[\frac{1}{2}\{n(n+1)\}\right]^2$;
therefore $1^3+2^3+3^3+\ldots n^3 = \left[\frac{1}{2}\{n(n+1)\}\right]^2$.

(3) In the identity
$$x^3 \equiv \left[\tfrac{1}{2}\{x(x+1)\}\right]^2-\left[\tfrac{1}{2}\{(x-1)x\}\right]^2,$$
write 1, 2, 3, ... n successively for x, and add.

5161. (EDITOR.)—In a given triangle place (1) a given circle so that the area of the triangle polar to the given triangle, relatively to the given circle, may be a minimum; and (2) show that, when the area of the polar triangle is constant, the locus of the centre of the circle is a cubic hyperbola.

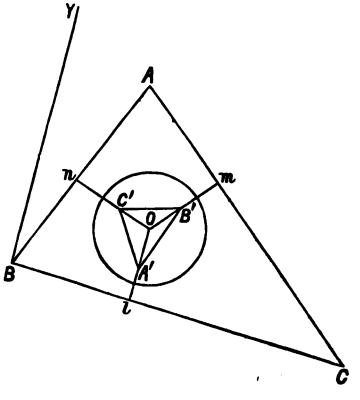
Solution by W. E. JEFFARES, B.A.

(1) Let O be the centre of the circle, A′, B′, C′ the poles of BC, CA, AB.

$$\frac{\triangle B'OC'}{\triangle ABC} = \frac{OB'.OC'}{bc} = \frac{r^4}{bc.cm.cn}$$

(A′B′C′ is the polar triangle of ABC with respect to the circle)

$$= \frac{r^4 \times \triangle BOC}{4\triangle AOB \times \triangle BOC \times \triangle COA} \quad \ldots (1).$$



Similarly,

$$\frac{\triangle C'OA}{\triangle ABC} = \frac{r^4\triangle COA}{4\triangle AOB \times \triangle BOC \times \triangle COA} \quad \ldots (2),$$

and
$$\frac{\triangle A'OB}{\triangle ABC} = \frac{r^4 \times \triangle AOB}{4\triangle BOC \times \triangle COA \times \triangle AOB} \quad \ldots\ldots\ldots\ldots (3).$$

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By adding (1), (2), (3), we have

$$\therefore \quad \frac{\triangle A'B'C'}{\triangle ABC} = \frac{r^4 \triangle ABC}{4 \triangle BOC \times \triangle COA \times \triangle AOB},$$

$$\triangle A'B'C' = \frac{r^4 (\triangle ABC)^2}{4 \triangle BOC \times \triangle COA \times \triangle AOB} \quad \ldots\ldots\ldots\ldots (4).$$

If $\triangle A'B'C'$ is a minimum, then

$$\triangle BOC \times \triangle COA \times \triangle AOB$$

is a maximum, since $r^4 (\triangle ABC)^2$ is constant.

Now, since the sum of triangles BOC, COA, AOB is given, their product is a maximum when they are equal, that is, when O is the centroid. Hence the centre O is a fixed point, and the radius of circle is given; therefore we can describe it.

(2) Take the base of the triangle as axis of x, and a perpendicular to it at B as that of y. Let fall from A a perpendicular AD; take $BD = s$, and $DC = s'$, $AD = p$.

$$\text{The area of BOC} = \tfrac{1}{2} \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ a & 0 & 1 \end{vmatrix} = \tfrac{1}{2} ay; \quad \therefore \quad 2 \triangle BOC = ay.$$

$$\text{The area of COA} = \tfrac{1}{2} \begin{vmatrix} x & y & 1 \\ a & 0 & 1 \\ s & p & 1 \end{vmatrix} = \tfrac{1}{2} \{ -px - (a-s) y + ap \}$$
$$-px - (a-s) y + ap = 2 \triangle COA.$$

$$\text{Area of AOB} = \tfrac{1}{2} \begin{vmatrix} x & y & 1 \\ s & p & 1 \\ 0 & 0 & 1 \end{vmatrix} = \tfrac{1}{2} (px - sy); \quad \therefore \quad 2 \triangle BOA = px - sy.$$

$$8 \triangle BOC \times \triangle COA \times \triangle AOB = ay (px - sy) \{ ap - px - (a-s) y \};$$

but $$8 \triangle BOC \times \triangle COA \times \triangle AOB = \frac{2r^4 \triangle ABC)^2}{\triangle' A'B'C'} = \text{constant} = \kappa^3,$$

$$ay (px - sy) \{ ap - px - (a-s) y \} = \kappa^3,$$
$$ay (px - sy) \{ ap - ay - (px - sy) \} = \kappa^3,$$
$$a^2 y (p-y)(px - sy) - a (px - sy)^2 y = \kappa^3, \quad \text{a cubic curve.}$$

[For another solution of this Question, see Vol. LXI., p. 63.]

13732. (Rev. W. Allen Whitworth, M.A.)—A point is taken at random on the area of (α) a circle; (β) a triangle; (γ) a square or any regular polygon; (δ) a rectangle, length $= a$, breadth $= b$; (ϵ) a parallelogram, longer side $= a$, shorter side $= b$, perpendicular breadth $= b'$. Show that the expectation of the shortest distance from the point to the boundary of the figure is (α) $\frac{1}{2}$ of radius; (β, γ) $\frac{1}{3}$ of radius of inscribed circle; (δ) $(3a-b) b \div 12a$; (ϵ) $(3a-b) b' \div 12a$.

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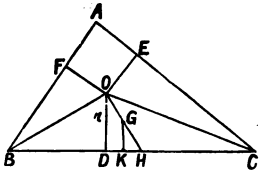
Solution by R. CHARTRES.

(1) The circle, triangle, and regular polygon may be taken together.

Let Δ_1 be the area of OBC, and G its centre of gravity; then expectation

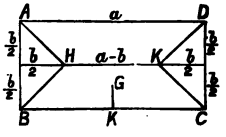
$$= \Sigma(\Delta_1/\Delta)\,GK = \tfrac{1}{3}r.$$



(2) The rectangle, parallelogram, and also the square. Bisect the angles A, B, &c., and let G be the centre of gravity of BCKH,

$$\text{expectation} = \frac{2BCKH}{ABCD}\,GK$$

$$= \frac{(3a-b)\,b}{12a}.$$

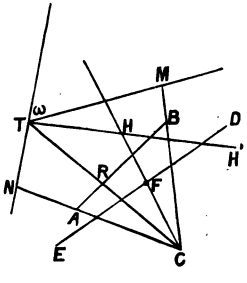


13721. (Professor NEUBERG.) — Connaissant deux points d'une parabole et un point de la directrice, trouver le lieu géométrique du foyer.

Solution by W. C. STANHAM, B.A.

Let A, B be the two points, C the point on the directrix, EFD the polar of C with respect to a parabola of the system, focus F. Then CF is perpendicular to DE.

If we reciprocate the figure with respect to C, and if A, B, and DE reciprocate into TN, TM, and H, F is the inverse with respect to C of the centre H of a rectangular hyperbola touching TN and TM and passing through C. Taking TN, TM as axes, the locus of H is readily found to be the curve



$$4x'y'\left\{2xy-\sec\omega\,(x^2+y^2)\right\}$$
$$=\left\{\sec\omega\,(x^2+y^2)+2xy-2y'x-2x'y\right\}^2,$$

which is the inverse with respect to T (radius k) of the conic

$$4x'y'\left\{2xy-\sec\omega\,(x^2+y^2)\right\}=\left\{\sec\omega k^2-2y'x-2x'y\right\}^2,$$

ω being the angle between the axes, and C the point (x', y'). Therefore the locus of the foci is an inverse with respect to C of a bicircular quartic which is an inverse with respect to T of a fixed conic (with known radii of inversion).

13498. (Professor H. LANGHORNE ORCHARD, M.A., B.Sc.)—If (in the usual notation) S denote the area of a plane triangle, and S_{abc} denote

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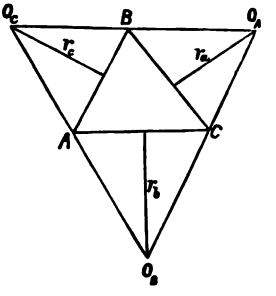
that of the triangle formed by joining the centres of the escribed circles, show that $S : S_{abc} = r : s$, s being the radius of the circumscribing circle.

Solution by Rev. J. CULLEN, B.A.; L. E. REAY, B.A.; *and* W. E. JEFFARES, B.A.

Let O_A, O_B, O_C be the ex-centres and σ the semi-perimeter of ABC.



Then $S_{abc} = S + \frac{1}{2}(ar_a + br_b + cr_c)$,

$r_a = S/\sigma - a$, &c.;

$\therefore\ S : S_{abc} = 2S^2 : \sigma abc$

$= r : 2R = r : s.$

For

$s = O_BO_C/2 \sin O_BO_AO_C$

$= (r_b + r_c)/2\cos^2 \frac{1}{2}A = 2R,$

since

$\angle BAO_C = \angle CAO_B = 90° - \frac{1}{2}A$, &c.

[Mr. CURJEL solves the Question as follows:—Using the usual notation,

$$AO_a = s/(\cos \tfrac{1}{2}A),$$

$$O_bO_c = S/(\cos \tfrac{1}{2}A)\left\{1/(s-c) + 1/(s-b)\right\} = Sa/\left\{(s-b)(s-c)\cos \tfrac{1}{2}A\right\};$$

$$\therefore\ S_{abc} = \tfrac{1}{2}\left[asS/\left\{(s-b)(s-c)\cos^2 \tfrac{1}{2}A\right\}\right] = \tfrac{1}{2}\left[(sSabc)/\left\{s(s-a)(s-b)(s-c)\right\}\right]$$

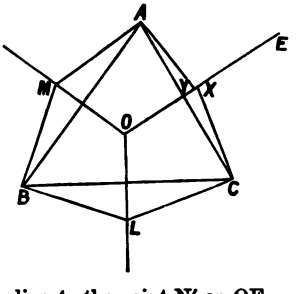
$$= sabc/2S = 2Rs;$$

$$\therefore\ S : S_{abc} = S : 2sR = r : 2R.]$$

13710. (J. WOODCOTE WEST.)—Through three points A, B, C to pass three circles, each circle to pass through two points and intersect the other circles orthogonally.

Solution by W. C. STANHAM, B.A.; *and* Professor SANJÁNA, M.A.

Through O, the circumcentre of ABC, draw OL, OE, OM perpendicular to the sides. Take any point X in OE; join CX; draw CL at right angles to CX, BM at right angles to BL, and AY at right angles to AM. Then it is clear that OE is divided homographically by corresponding points X, Y, and the double points of the system give two solutions of the problem. For, if N′ is a double point, the circles with centres L′, M′, N′ and radii CL′, BM′, AN′ fulfil the required conditions, L′ and M′ being the points on OL and OM corresponding to the point N′ on OE.



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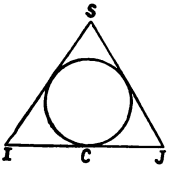
13738. (R. F. Davis, M.A.)—AB is a fixed chord of a given conic, and PQ a variable chord having a fixed direction. Prove that the foci of the parabolas through APQB lie on two fixed conics.

Solutions by (1) W. J. Dobbs, M.A., *and* Rev. J. Cullen, B.A.;
(2) W. C. Stanham, B.A.

(1) Project the points A, B into the circules. Then the given conic becomes a circle, and the circules become real points I, J. Also PQ meets IJ in a fixed point D.

The parabola becomes a circle touching IJ at a point C, where

sq. on DC = rect. DP . DQ = const.



Hence the point C has one of two fixed positions.

We have now to determine the following locus:

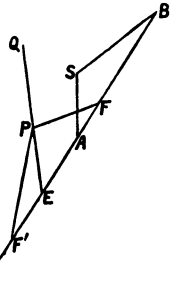
A variable circle touches a fixed line IJ *at a fixed point* C, *and tangents from the fixed points* I *and* J *meet at* S: *it is required to determine the locus of* S.

It is easily seen that SI ~ SJ = CI ~ CJ = const. Hence the locus of S is a conic with foci at I and J.

Hence, returning to the original problem, and remembering that C has one of two fixed positions, we see that the foci of the parabolas lie on two fixed conics with foci at A and B.

(2) Let PQ denote any position of the variable chord, S the focus of a parabola through A, B, Q, P, the axis of which is parallel to PF. Then, as is well known, if BA, QP meet at E,

$$EF^2 : EP^2 = EA \cdot EB : EP \cdot EQ.$$



And, since the directions of AB and QP are fixed, the latter ratio is constant in any conic. Also the angle PEF is constant. Therefore the angle PFE is constant, and consequently the difference of the distances of B and A from the directrix of the parabola is constant, or SB ~ SA is constant. Therefore the locus of S is a hyperbola having A and B for foci. Similarly, if F′ is taken in BA produced, such that EF′ = EF, it is clear that the focus of the other parabola through A, B, Q, P lies on a confocal hyperbola.

13706. (W. E. Jeffares, B.A.)—A triangle is formed by two tangents to a parabola and their chord of contact. Prove that the line joining the intersection of the tangents to the focus is the symmedian of this triangle, and, if produced to meet the circumcircle, will be bisected at the focus.

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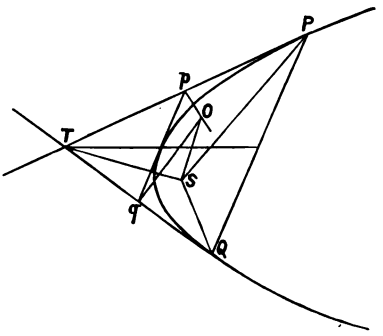
Solution by H. W. Curjel, M.A.; *and* L. E. Reay, B.A.

Let TP, TQ be the tangents; V, p, q the middle points of PQ, TP, TQ.



Then $\angle QTS = SPT$

$= VTP.$

Therefore TS is a symmedian of $\triangle PQT$.

Also circle Tpq passes through S, and OT is clearly a diameter.

Therefore OST is a right angle.

Therefore, if TS is produced to meet circle PTQ, it will be bisected at S.

4669. (F. A. Tarleton.)—Find the locus of the intersection of the perpendiculars of a triangle inscribed in one conic, and circumscribed about another.

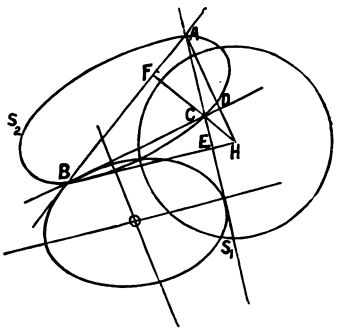
Solution by Professor Sanjána, M.A.

Let the triangle ABC be circumscribed to

$$S_1 \equiv x^2/a^2 + y^2/b^2 - 1 = 0,$$

and inscribed in

$$S_2 \equiv c^2x^2 + d^2y^2 + e + 2fy + 2gx + 2hxy = 0.$$



Let the coordinates of H, the orthocentre, be X, Y; and let

$$HD \cdot HA = HF \cdot HC = HE \cdot HB = \rho^2;$$

then

$$S_3 \equiv (x-X)^2 + (y-Y)^2 - \rho^2 = 0$$

is the circle of which ABC is the self-conjugate (autopolar) triangle.

As the triangle circumscribing S_1 is autopolar to S_3, the invariant Θ for these is zero; this gives

$$-(1/b^2) - (1/a^2) + \{(X^2 + Y^2 - \rho^2)/a^2b^2\} = 0; \quad \therefore \quad \rho^2 = X^2 + Y^2 - a^2 - b^2.$$

As the triangle inscribed in S_2 is autopolar to S_3, the invariant Θ' for these is zero; this gives

$$(X^2 - \rho^2)\, c^2 + (Y^2 - \rho^2)\, d^2 + e + 2Yf + 2Xg + 2XYh = 0;$$

therefore $\rho^2 (c^2 + d^2) = c^2X^2 + d^2Y^2 + e + 2fY + 2gX + 2hXY.$

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Eliminating ρ^2 and employing small letters, we see that the locus of H is

$$S_2 = (c^2+d^2)(x^2+y^2-a^2-b^2),$$

a conic having axes parallel to those of S_2. [SALMON'S *Conics*, sixth edition, pp. 340–342.]

13748. (C. E. BICKMORE, M.A.)—Show that

$$\begin{vmatrix} x, & y, & z, & w \\ -w, & x, & y, & z \\ -z, & -w, & x, & y \\ -y, & -z, & -w, & x \end{vmatrix} = a^2+b^2 = c^2+2d^2 = e^2-2f^2,$$

a, b, c, d, e, f being definite quadratic functions of x, y, z, w.

Solution by the PROPOSER.

Multiplying each constituent of the second column by ρ, the third column by ρ^2, and the fourth column by ρ^3, and adding to the corresponding constituents of the first column, the latter becomes

$$x+y\rho+z\rho^2+w\rho^3, \quad x\rho+y\rho^2+z\rho^3-w, \quad x\rho^2+y\rho^3-z-w\rho, \quad x\rho^3-y-z\rho-w\rho^2.$$

Hence, if $\rho^4 = -1$, $x+y\rho+z\rho^2+w\rho^3$ is a factor of the determinant, which is consequently the product of

$$(x+y\rho+z\rho^2+w\rho^3)(x-y\rho+z\rho^2-w\rho^3)(x+y\rho^3-z\rho^2+w\rho)(x-y\rho^3-z\rho^2-w\rho)$$

by some numerical constant, easily seen to be 1, since x^4 is the principal element. Hence

$$\begin{aligned} \text{determinant} &= \{(x+z\rho^2)^2-(y\rho+w\rho^3)^2\}\{(x-z\rho^2)^2-(y\rho^3+w\rho)^2\} \\ &= (x^2+2yw-z^2)^2+(y^2-2xz-w^2)^2. \end{aligned}$$

Similarly, combining the four factors in two other ways,

$$\begin{aligned} \text{determinant} &= (x^2-y^2+z^2-w^2)^2+2\{x(y+w)-z(y-w)\}^2 \\ &= (x^2+y^2+z^2+w^2)^2-2\{x(y-w)+z(y+w)\}^2; \end{aligned}$$

$$\therefore\ a = x^2+2yw-z^2, \quad c = x^2-y^2+z^2-w^2, \quad e = x^2+y^2+z^2+w^2,$$
$$b = y^2-2xz-w^2, \quad d = xy+xw-yz+zw, \quad f = xy-xw+yz+zw.$$

[The determinant is the norm of the octavic integer $x+y\rho+z\rho^2+w\rho^3$, and the quantities (a, b), (c, d), (e, f), (x, y, z, w) have important bearings on the theory of biquadratic and octavic residues.]

9082. (Professor BORDAGE.)—Prove that, (1) in a right-angled triangle $\cos(2C-B) = ca^{-3}(3c^2-4a^2)$; and, reciprocally, (2) if in a triangle we have one of the relations

$$\sin C-\cos B = \cos A, \quad 1+(\cot B-1)^{-1} = (1-\cot C)^{-1},$$

he triangle is right-angled.

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Solution by H. J. WOODALL, A.R.C.S.

(1) $\cos(2C-B) = \cos 2C \cos B + \sin 2C \sin B$
$= (2\cos^2 C - 1)\cos B + 2 \sin C \cos C \sin B.$

Substitute $= \{(a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2)/2a^2b^2-1\}(a^2+c^2-b^2)/2ac$
$+2\{2S(a^2+b^2-c^2)\,4S\}/(ab\ 2ab\ 2ac)$
$= 1/(4a^3b^2c)\{(2a^2b^2-16S^2)(a^2+c^2-b^2)+16S^2(a^2+b^2-c^2)\}$
$= 1/(4a^3b^2c)\{2a^2b^2(a^2+c^2-b^2)+32S^2(b^2-c^2)\}.$

Put $a^2 = b^2+c^2$; therefore $2S = bc$; this becomes
$c/a^3(3a^2-4c^2) = ca^{-3}(3a^2-4c^2).$

(2) $\sin C-(\cos B+\cos A) = 2\sin\frac{1}{2}C\{\sin\frac{1}{2}(A+B)-\cos\frac{1}{2}(A-B)\}$
$= 2\sin\frac{1}{2}C(\cos\frac{1}{2}B-\sin\frac{1}{2}B)(\sin\frac{1}{2}A-\cos\frac{1}{2}A);$

therefore vanishes if A or B $= \frac{1}{2}\pi$, but not if C $= \frac{1}{2}\pi$.

$$1+(\cot B-1)^{-1}+(\cot C-1)^{-1}=0;$$

multiply through by $(\cot B-1)(\cot C-1)$, and reduce, when we get

$$\cot B \cot C = 1,$$

giving $B+C = \frac{1}{2}\pi$, whence $A = \frac{1}{2}\pi$; hence the angle A is right.

13094. (Rev. J. CULLEN, B.A.)—Evaluate the following integral:—

$$\int_0^{\pi} \frac{\sin^2 3\theta\, d\theta}{1+e^{a\cos\theta}}.$$

Solution by Professors SANJÁNA, FINKEL, *and others.*

Let I denote the integral; then

$$I = \int_0^{\pi} \frac{\sin^2 3(\pi-\theta)\,d\theta}{1+e^{a\cos(\pi-\theta)}} = \int_0^{\pi} \frac{\sin^2 3\theta\, d\theta}{1+e^{-a\cos\theta}};$$

$$\therefore\quad 2I = \int_0^{\pi} \sin^2 3\theta \left(\frac{1}{1+e^{a\cos\theta}}+\frac{1}{1+e^{-a\cos\theta}}\right)d\theta = \int_0^{\pi}\sin^2 3\theta\, d\theta.$$

Thus
$$I = \tfrac{1}{2}\int_0^{\pi}\sin^2 3\theta\, d\theta = \tfrac{1}{4}\int_0^{\pi}(1-\cos 6\theta)\,d\theta$$
$$= \tfrac{1}{4}\int_0^{\pi} d\theta - \tfrac{1}{4}\int_0^{\pi}\cos 6\theta\, d\theta = \tfrac{1}{4}\pi.$$

13728. (D. BIDDLE.)—It is well known that all square numbers are of the form $5n$ or $5n\pm1$, and that all cube numbers are of the form $7n$ or $7n\pm1$. Prove that all *prime* numbers (above 3) are of the form $6n\pm1$, and find the law of exception to the converse.

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Solutions (1) *by* Rev. J. CULLEN, B.A.; (2) *by* R. W. D. CHRISTIE.

(1) Let N be any odd number prime to 3.

Then $N^2-1 = M(3) = M(6)$, since N^2-1 is even;

therefore $N-1$ or $N+1$ is divisible by 6; therefore $N = 6n \pm 1$.

Hence, every prime number is of this form, and also every odd number prime to 3.

(2) Primes greater than 3 are of form $2n+1$, and also of form $3n \pm 1$; otherwise they would be divisible by 2 or by 3; therefore $P = 2n+1 = 3m \pm 1$.

(i.) $2n+1 = 3m+1$; therefore $2n = 3m = 6p$;

thus $n = 3p$ and $P = 6p+1$.

(ii.) $2n+1 = 3m-1$; therefore $2(n+1) = 3m = 6p$;

$m = 2p$ and $P = 6p-1$.

Again, all numbers are of *one* of the forms $6m$, $6m \pm 1$, $6m \pm 2$, $6m \pm 3$; therefore the only forms permissible for primes are $6m \pm 1$.

If $6m - ab = \pm 1$ where ab is a composite number greater than 3, we must have m prime to both a and b, and a and b prime to 6.

The cases may be distinguished thus—

If $ab - 6m = 1$,

then $a = 6r-1$, $b = 6s-1$ or $a = 6r+1$, $b = 6s+1$.

If $6m - ab = 1$,

then $a = 6r-1$, $b = 6s+1$.

Editorial Note to 13728.

[As to the law of exception to the converse proposition, namely, that numbers of the form $6n \pm 1$ are prime, it is easy to see that when numbers thus formed are composite the factors must be of somewhat similar form, as shown by Mr. CHRISTIE. Thus, in the exceptional cases,

$$6n+1 = (6p+1)(6q+1), \text{ or } (6p-1)(6q-1) \ldots\ldots (\alpha), (\beta),$$

$$6n-1 = (6p+1)(6q-1), \text{ or } (6p-1)(6q+1) \ldots\ldots (\gamma), (\delta).$$

The factors themselves may be similarly composite, or they may be prime. By multiplication of the factors we have

$$6n+1 = 6\{6pq \pm (p+q)\} + 1 \ldots\ldots\ldots\ldots (A), (B),$$

$$6n-1 = 6\{6pq \pm (p-q)\} - 1 \ldots\ldots\ldots\ldots (C), (D).$$

It is further easy to see that n is represented by the value within brackets in these four cases, and that, if p and q are integral, whilst such an equation holds good, the number under investigation is composite and not prime. In order to show how any of these values of n may be utilized for purposes of trial, we will take the case C, in which $n = 6pq - (p-q)$, and $p > q$. Let $p-q = x$; then $pq = (n+x)/6$, and, since pq is

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integral, $n+x$ must be a multiple of 6, and n and x are odd or even together—a fact to be noted for use hereafter. Let $(n/6)^{\frac{1}{2}} = S'+s$, in which S' is the integer and s the decimal. Then, if we assume that $x \equiv (p-q)$ may be of any value above zero, and that k' may be found by trial, such that $2S'+k' = (p+q)$, also integral, we have

$$(2S'+k')^2 - \tfrac{2}{3}(n+x) = x^2,$$

which represents $(p+q)^2 - 4pq = (p-q)^2$, a modification of a well-known formula, and we can now obtain the equation

$$3x+1 = \left\{9(2S'+k')^2 - N\right\}^{\frac{1}{2}},$$

in which $N = 6n-1$, the original number. It is easy to see that $p-q$, $p+q$, and also k' are odd or even together, that is, with n. Consequently, in making trial, we take k' as 1, 3, 5, &c., when n is odd; but as 2, 4, 6, &c., when n is even. The equations for the four cases are as follows:—

(A), (B) ..... $3x = \left\{\left[3(2S'+k')\pm 1\right]^2 - N\right\}^{\frac{1}{2}}$;

(C), (D) ..... $3x \pm 1 = \left\{9(2S'+k')^2 - N\right\}^{\frac{1}{2}}$.

The object, of course, is by manœuvring k' to make the value within brackets $= \square$. By this method we can split up 8616460799, the number mentioned by Jevons as impracticable (*Nature*, Feb. 28, 1889), into its two (prime) factors after not more than 10 trials, that is, when $k' = 20$.

But, as affording an alternative, and (where N is small) a much simpler method, it is worthy of mention that in the four cases above given, the following values are all integral:—

in (A) ..... $(n-p)/q = 6p+1$, $(n-q)/p = 6q+1$;

in (B) ..... $(n+p)/q = 6p-1$, $(n+q)/p = 6q-1$;

in (C) ..... $(n+p)/q = 6p+1$, $(n-q)/p = 6q-1$;

in (D) ..... $(n-p)/q = 6p-1$, $(n+q)/p = 6q+1$;

On the dexter of each pair of equations we have the factors of N, and on the sinister a value which (being necessarily integral) can often be found by the simplest inspection. Thus, in factorizing 559, we have $n = 93$, and the case is (A) or (B); $(n/6)^{\frac{1}{2}}$ does not exceed 4. Below this the lesser factor of pq must lie. $(93-2)/7$ and $(93-7)/2$, where, be it observed, 7 and 2 simply change places, are both integral; and we have 13 and 43 as factors of N. But the instance is one which shows that caution is needed; for we might have taken $(93-2)/13$ and $(93-13)/2$, which are also both integral; but the quotient in the last-mentioned, namely, 40, is not of the form $6p \pm 1$, and therefore must be discarded. Besides which, we already know that 2.7, as representing pq, is much nearer $n/6$ than is 2.13. And further help is afforded by the fact that, if we take $n = 6T+R$, where $R < 6$, $p+q$ in cases A, B, and $p-q$ in cases C, D, will be represented by $R+6r$ or $6r-R$, according as $n-q$ or $n+q$ is the numerator in $(n \pm q)/p$; r being 0, 1, 2, &c. In the present instance, $R = 3$, $r = 1$, and $p+q = R+6 = 9$; therefore, having taken $q = 2$, we take $p = 7$. A small table (to S′ terms), made beside our work, of successive values of $R+6r$ and $6r-R$ is of great use.]

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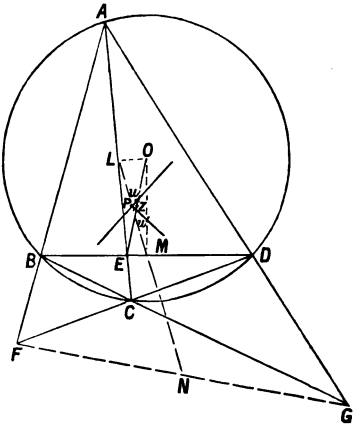
13773. (R. F. Davis, M.A.)—Let A, B, C, D be four points on a circle, centre O; P their mid-centre; and E, F, G the intersections of (AC, BD), (BA, CD), (AD, BC) respectively. Prove (1) the axes of the two parabolas through ABCD intersect at right angles in P, which point lies on the nine-point circle of EFG; (2) the directrices intersect in the circumcentre of EFG; (3) if Z be the mid-point of OE, and ZU, ZU′ be taken along DE, such that ZU = ZU′ = ZP, then UP, U′P intersect the nine-point circle of EFG again in the two foci S, S′.

Solution by Professor A. Droz-Farny.



On sait que, si une parabole est coupée par une circonférence, le barycentre des quatre points d'intersection se trouve sur l'axe de la parabole. Si en outre par quatre points d'une circonférence on fait passer un faisceau de coniques, toutes ces courbes ont leurs axes parallèles aux bissectrices des angles que forment les couples de côtés opposés du quadrilatère. Donc les axes des deux paraboles se coupent orthogonalement en un point P, barycentre des quatre sommets A, B, C, D. Ce point P, intersection des trois droites qui joignent les points milieux des paires de côtés opposés du quadrilatère, ainsi que de la droite qui joint les points milieux des deux diagonales, est en outre le centre de l'hyperbole équilatère, lieu géométrique des centres des coniques circonscrites au quadrilatère ABCD; elle passe par les six points milieux déjà cités, par le centre O par les sommets E, F, G. Son centre P doit donc se trouver sur le cercle des neuf points de ce triangle. Voici d'ailleurs une démonstration directe de ce dernier fait.

Soient L, M, N les points milieux des diagonales AC, BD, FG. Ces trois points sont en ligne droite. Le triangle EFG étant conjugué par rapport au cercle, O est son orthocentre, et par conséquent la circonférence décrite sur ZN comme diamètre est le cercle des neuf points du triangle EFG. Or dans le quadrilatère inscriptible LOEM, P étant le milieu de LM, ZP sera perpendiculaire sur PN, donc le cercle des neuf points passe par P.

Le triangle EFG étant conjugué par rapport à toutes les coniques du faisceau ABCD, d'après le théorème de Faure, son cercle circonscrit est orthogonal à tous les cercles de Monge des coniques du faisceau; donc aussi aux directrices des deux paraboles qui seront par conséquent des diamètres de cette circonférence. En outre le triangle complémentaire de EFG sera circonscrit aux deux paraboles et par conséquent le cercle des neuf points de EFG contiendra les foyers S et S′ de ces deux courbes.

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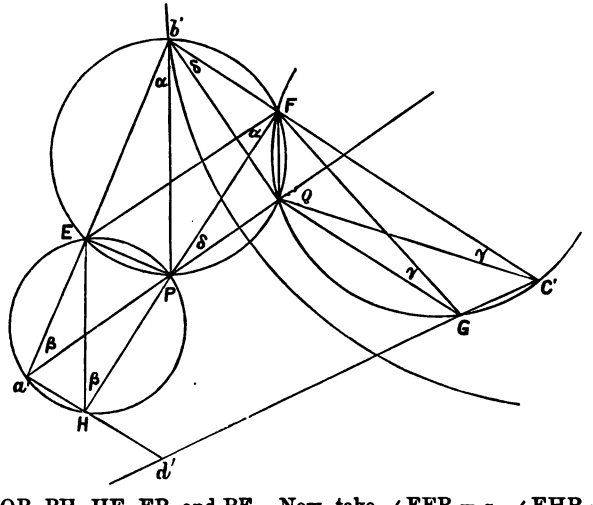
Les axes des paraboles sont les asymptotes de l'hyperbole équilatère précédemment citée ; pour les construire comme P est le centre de cette courbe à laquelle appartiennent les points O et E en portera sur OE, ZU = ZU′ = ZP ; PU et PU′ seront les asymptotes, car elles se coupent à angle droit et elles sont isotomiques par rapport aux points O et E. Les deux points S et S′ suivant lesquelles elles rencontrent le cercle dss neuf points seront dont les foyers des paraboles.

13485. (Charles Mitchell.)—*abcd* is a given quadrilateral ; E, F, G, and H are four given points. Draw through these points a quadrilateral, having its sides proportional and similarly placed, and its angles equal and similarly placed with those of *abcd*. The sides proportional to *ab*, *bc*, *cd*, *da* to pass respectively through E, F, G, and H.

Solution by W. E. Jeffares, B.A.

On EF describe a segment of circle containing an angle equal to *abc*, and on FG and HE segments containing angles equal to the angles *bcd* and *dab* of the given quadrilateral *abcd*; and let the last two circles cut that on EF in Q and P ; then Q and P are fixed points. Join EF, FG,



GQ, QP, PH, HE, EP, and PF. Now, take ∠EFP = α, ∠EHP = β, ∠FPQ = δ, ∠FGQ = γ (which are all fixed angles). Now find the locus of the vertex of all triangles on PQ so that the sides

$$Pb' : Qb' :: ab \sin(\gamma + \delta) \sin\beta : bc \sin\gamma \sin(\alpha + \beta).$$

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Let this circle intersect that on EF in b'; therefore b' is fixed. Join b' with F, and produce it to meet the circle FQG in c'; also, join b' with E, and produce it to meet the circle PHE in a'. Again, join a' and H and c' with G, and then produce them to meet in d'. The quadrilateral $a'b'c'd'$ is the required one.

$$\frac{Pb'}{a'b'} = \frac{\sin\beta}{\sin(\alpha+\beta)}, \quad \frac{Qb'}{b'c'} = \frac{\sin\gamma}{\sin(\gamma+\delta)};$$

therefore $$\frac{a'b'}{b'c'} \times \frac{Qb'}{Pb'} = \frac{\sin\gamma\sin(\alpha+\beta)}{\sin\beta\sin(\gamma+\delta)}, \quad \frac{a'b'}{b'c'} = \frac{Pb'}{Qb'}\frac{\sin\gamma\sin(\alpha+\beta)}{\sin\beta\sin(\gamma+\delta)},$$

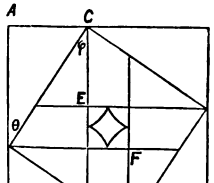
but $$\frac{Pb'}{Qb'} = \frac{ab}{bc}\frac{\sin\beta\sin(\gamma+\delta)}{\sin\gamma\sin(\alpha+\beta)}; \qquad \text{(by constr.)}$$

therefore $$\frac{a'b'}{b'c'} = \frac{ab}{bc}.$$

Now, since $a'b' : b'c :: ab : bc$, therefore the triangle $a'b'c'$ is similar to the triangle abc, and the triangle $a'c'd'$ is similar to acd; therefore the quadrilaterals $a'b'c'a'$ and $abcd$ are similar.

12706. (Professor SHIELDS.)—There is a large *square* piece of land AB in which is laid off another *square* CD, whose sides are oblique to the sides of the large square; and the corners of the inner square touch the sides of the large square at a distance equal to one-third of their length from the corners; the inner square, being divided by lines drawn from each of the four corners to the middle of the opposite sides, forms another square EF in the centre. These four lines and sides of the centre square are parallel to the sides of the large square. From each corner of the centre square EF, *as centres*, are drawn four equal quadrants tangent to each other, and thus enclosing $2\frac{73}{500}$ acres of land in the centre G. Find the size and area of each square, and the area of *one* quadrant.

Solution by Rev. J. L. KITCHIN, M.A.; Professor SARKAR; *and others.*



The squares of two sides equal square on diameter. Hence it is a square; $\tan\theta = \frac{1}{2}$, so also $\tan\phi = \frac{1}{2}$; therefore sides of innermost square are parallel to sides of square AB.

Let x = side of outer square; then

$\frac{1}{3}\sqrt{5}\,x$ = side of CD, $\frac{1}{3}x$ = side of EF.

The four quadrants form a complete circle radius $\frac{1}{6}x$; therefore inner area

$$= \tfrac{1}{9}x^2 - \tfrac{1}{36}\pi x^2 = \tfrac{1}{36}(4-\pi)x^2$$
$$= \tfrac{1073}{500} \times 4840 \text{ square yards};$$

therefore $$\tfrac{8584}{10000} \cdot \tfrac{1}{36}x^2 = \tfrac{1073}{50} \times 484,$$

$$\tfrac{8}{10000} \cdot \tfrac{1}{36}x^2 = \tfrac{484}{50}, \quad \tfrac{4}{100} \cdot \tfrac{1}{6}x^2 = \tfrac{22}{5}; \quad \therefore \quad x = 660 \text{ yards};$$

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side of CD $= 220\sqrt{5}$, side of EF $= 220$;

therefore square AB $= [(660)^2/4840]$ acres $= 90$,

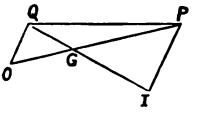
square CD $= \frac{5}{9}$ of AB $= 50$ acres,

square EF $= \frac{1}{9}$ of AB $= 10$ acres,

and one quadrant $= 1{\cdot}9625$ acres.

5365. (Christine Ladd.)—If P be the orthocentre of a triangle, I the centre of the incircle, Q the centre of the circle inscribed in the orthocentric triangle (*i.e.*, the triangle formed by joining the mid-points of the sides), prove that $QP^2 - QI^2 = 2(Rr - R\rho - r^2)$.

Solution by Professor K. J. Sanjána.



Let G be the common centroid of the triangle, and its *orthocentric* triangle; then it is the centre of similitude of the two. Make $GO = \frac{1}{3}GP$; then O is the circumcentre of the triangle. Make $GQ = \frac{1}{3}GI$; then Q is the incentre of the *orthocentric* triangle. Join QP, QO. It is easily seen that $GP^2 + 2QO^2 = 3QG^2 + PG^2 + 2OG^2$, and $QI^2 = \frac{2}{3}GI^2$; hence

$$QP^2 - QI^2 = \tfrac{1}{3}OP^2 - \tfrac{1}{3}GI^2 - \tfrac{1}{3}PI^2$$

$$= \tfrac{1}{3}R^2(1 - 8\cos A\cos B\cos C) - 8R^2(\Sigma \sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C - \tfrac{1}{16}\Sigma\sin^2 A)$$
$$- 2R^2(8\sin^2\tfrac{1}{2}A\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C - \cos A\cos B\cos C)$$

$$= \tfrac{1}{3}R^2\{1 - 8\cos A\cos B\cos C - 3\Sigma(1-\cos B)(1-\cos C) + \Sigma\tfrac{1}{2}(1-\cos 2A)$$
$$- 3(1-\cos A)(1-\cos B)(1-\cos C) + 3\cos A\cos B\cos C\}$$

$$= \tfrac{1}{3}R^2\{-9 + 9\cos A + 9\cos B + 9\cos C - 6\cos B\cos C - 6\cos C\cos A$$
$$- 6\cos A\cos B\}$$

$$= 2R^2\{-1 + \cos A + \cos B + \cos C - 2 + 2\Sigma\cos A - 2\Sigma\cos B\cos C$$
$$+ 2\cos A\cos B\cos C - 2\cos A\cos B\cos C\}$$

$$= 2R^2\{4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C - 2\,.\,2\sin^2\tfrac{1}{2}A\,.\,2\sin^2\tfrac{1}{2}B\,.\,2\sin^2\tfrac{1}{2}C$$
$$- 2\cos A\cos B\cos C\}$$

$$= 2\{Rr - r^2 - R\rho\},$$

where ρ is the radius of the incircle of the *pedal* triangle.

13655. (W. C. Stanham.)—a and b are the semiaxes of an ellipse; $c^2 = a^2 - b^2$, $d^2 = a^2 + b^2$. From a point T on the director circle the tangents are drawn, and the corresponding normals meet at N. Show (1) that N lies on the curve

$$b^2(c^2 - dr)^2 a^2(c^2 + dr) = \tan^2\theta$$

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(the centre and major axis being the pole and initial line); (2) N lies within the evolute of the ellipse; (3) if the tangents to the ellipse at the feet of the other two normals from N meet at T', T' lies outside the rectangle formed by the tangents at the vertices of the ellipse, and between the tangents at opposite vertices.

Solution by the PROPOSER.

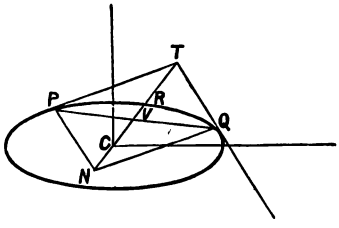
(1) Denote CN and CR by r and r_1. Then

$$r + CV = QV,$$
$$CV\,d = d^2 - r_1^2,$$
$$QV^2 : r_1^2 - CV^2 = d^2 - r_1^2 : r_1^2,$$

whence $d(d-r) = 2r_1^2$,

and $r_1^2(\cos^2\theta/a^2 + \sin^2\theta/b^2) = 1$.

Eliminating r_1, we obtain the equation in (1).



(2) The equation of a normal is of the form $(my + x)(a^2m^2 + b^2)^{\frac{1}{2}} = (a^2 - b^2)m$.

Therefore for the normals through $N(x_1, y_1)$, two of which are at right angles, the roots of the biquadratic in m are of the form m, $-1/m$, m_1, m_2, and therefore $m_1m_2 = -b^2x_1^2/a^2y_1^2$, a negative quantity, so that m_1 and m_2 must be real and unequal. Therefore (2) follows.

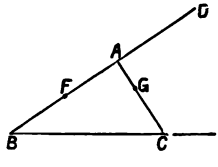
(3) The inclination of the tangents through $T'(x', y')$ are given by $(y' - mx')^2 = a^2m^2 + b^2$, and, since these are at right angles to the normals given by m_1 and m_2, in (2) it is clear that $(y'^2 - b^2)/(x'^2 - a^2)$ must be negative, which proves (3).

9367. (F. MORLEY, M.A.)—In the sides AB, AC of a triangle ABC, find points D, E such that BD = DE = EC.

Solution by I. ARNOLD.

Let ABC be a triangle right-angled at A; find two points F and G, so that

$$CG = GF = FB.$$



Analysis.—Let

$$AC = b, \quad AB = c, \quad \text{and} \quad GC = x;$$

then $(b-x)^2 + (c-x)^2 = x^2$,

or $b^2 + c^2 - (2b + 2c)x + 2x^2 = x^2$;

or $x^2 - 2(b+c)x = -(b^2 + c^2)$, or $x = (b+c) - \sqrt{(2bc)}$.

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Hence we have the following construction. Produce BA to D, making AD = AC. Let DF be the side of a square equal to twice the rectangle AB, AC; make CG = BF, and join GF; then CG = GF = BF. This is only a particular solution.

[For other Solutions of this Question, see Vol. XLIX., p. 51, and Vol. LI., p. 45.]

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797. (MORTIMER COLLINS.)—A uniform heavy chain 1000 feet long is hung from two points in the same horizontal plane 985 feet asunder; find the tension at the lowest point of the curve.

Solution by Professor G. B. M. ZERR.

Let $2l$ = length of chain.

$2a$ = distance asunder of the two points,

$s = \frac{1}{2}c\,(e^{x/c} - e^{-(x/c)})$ be the equation to the curve.

m = unit weight of chain;

$\therefore$ tension required $= mc = wc/2l$ where w = weight of chain,

$$l = \tfrac{1}{2}c\,(e^{a/c} - e^{-(a/c)}) = \tfrac{1}{2}c\left\{\frac{2a}{c} + \frac{2a^3}{3!\,c^3} + \frac{2}{5!}\frac{a^5}{c^5} + \frac{2}{7!}\frac{a^7}{c^7} + \frac{2}{9!}\frac{a^9}{c^9} + \ldots\right\}$$

$$= a + \frac{1}{3!}\frac{a^3}{c^2} + \frac{1}{5!}\frac{a^5}{c^4} + \frac{1}{7!}\frac{a^7}{c^6} + \frac{1}{9!}\frac{a^9}{c^8} + \ldots;$$

$$\therefore \quad l - a = \frac{1}{3!}\frac{a^3}{c^2} + \frac{1}{5!}\frac{a^4}{c^5} \text{ approximately.}$$

$$c = \frac{a}{\{12(l-a)\}^{\frac{1}{4}}}\left\{a + \left(\frac{a\,(6l-a)}{5}\right)^{\frac{1}{4}}\right\}^{\frac{1}{4}}; \text{ but } l = 500,\ a = 492\tfrac{1}{2},\ c = 1633{\cdot}13.$$

T = tension required = $1{\cdot}63313w$ nearly.

13716. (W. S. COONEY.)—Squares are described on the sides of any triangle ABC (same figure as 13568). If T = area of triangle formed by intersections of AA′, BB′, CC′, T′ = area of triangle formed by the whole lines AA′, BB′, CC′, S = area of triangle formed by intersections of A″B′, B″C′, C″A′, and S′ = area of triangle formed by the whole lines A″B′, B″C′, C″A′, prove (1) that the four triangles are similar, and that $T/S = T'/S' = \frac{1}{2}$, the orthocentre of triangle formed by joining centres of squares (the point O of Quest. 13481) being a double point of T and S; (2) that the distance of any angle A of the triangle from centre of opposite square equals distance between the centres of the other two squares.

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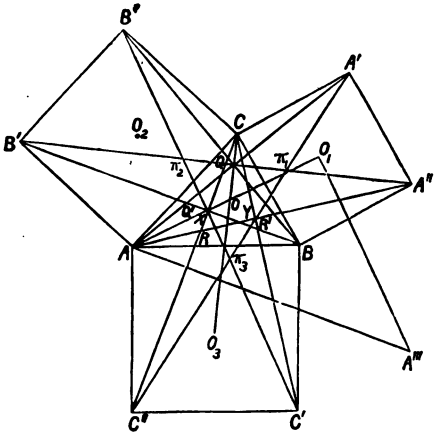
Solution by the PROPOSER.

AA′, BB′, CC′	intersect in	P′Q′R′	(T),
AA″, BB″, CC″	,, ,,	PQR	(T),
A″B′, B″C′, C″A′	,, ,,	$\pi_1\pi_2\pi_3$	(S),
BB′, CC″, B″C′, AO_1	,, ,,	X,	
CC′, AA″, C″A′, BO_2	,, ,,	Y.	



If AA‴ be drawn parallel to B′B, the $\Delta AA'A''' = \Delta$ formed by AA′, BB′, CC′ (T′), and is similar to P′Q′R′ or PQR.

$\Delta AO_1A'''$ is right-angled at O_1, and $AO_1 = A'''O_1$.

If B‴ and C‴ be found in same way, $\Delta A'''B'''C''' = \Delta$ formed by A″B′, B″C′, C″A′ (S′), which is similar to $\pi_1\pi_2\pi_3$.

Again, $2AO_1 = 2A'''O_1 = A'''C''' = B''C' = 2O_2O_3$; $\therefore AO_1 = O_2O_3$,

and, since $A'''C''' = 2AO_1$ and $BB' = \sqrt{2}\,AO_1$, $\therefore A'''C''' = \sqrt{2}\,BB'$;

therefore $S' = 2T'$, the triangles being similar.

All the angles at X and Y are half right angles; therefore

$$\angle XRY = X\pi_3Y;$$

therefore $OXR\pi_3R'Y$ is cyclic, and $O\pi_3$ the diameter, and, since $O\pi_3 = \sqrt{2}\,OR$, if ΔPQR be turned round O through half a right angle,

$$\Delta\pi_1\pi_2\pi_3 = 2\Delta PQR; \quad \therefore S = 2T; \quad \therefore T/S = T'/S' = \tfrac{1}{2}.$$

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103. (EDITOR.)—Through a given point P within a given angle ACB draw a straight line that (1) the triangle ABC shall be a minimum; or (2) the intercepted part APB shall be a minimum. [Solutions are given

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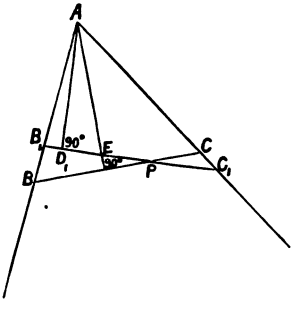
by Mr. WOODALL and Professor LAMPE to the second part. In CASEY'S *Sequel*, Bk. III. (18), the converse proposition is proved, but a simple direct proof is desired.]

Solution by R. CHARTRES.



Let $BC = a = B_1C_1$ for two consecutive positions of the minimum line through P, and let

$$PC = x, \quad PB = y,$$
$$PD = z, \quad DE = t;$$

then $\frac{\Delta ABC - \Delta AB_1C_1}{\Delta PDE}$ ultimately

$$= at/zt = a/z;$$

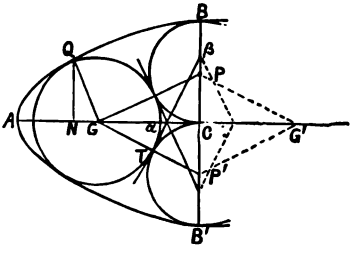
but this also $= \frac{y^2-x^2}{z^2} = \frac{a(y-x)}{z^2}$;

$\therefore \quad z = y-x, \quad$ or $\quad PC = BD.$

[Other Solutions of this Question are given in Vol. LXI., p. 25.]

4959. (S. WATSON.)—Two circles (each radius r) are described on the semi-minor axes of an ellipse as diameters, and two others (each radius r') having their centres on the major axis to touch the former two and the ellipse, A is the area of the rhombus formed by joining the centres of the four circles, and A′ the area of that formed by tangents at the points of contact of the circles; then $r'/r = 2e^2$, $A/A' = 1 + 1/e^2$.

Solution by Professor SANJÁNA.



Let P, P′ be the centres of the circles on CB, CB′, so that $r = b/2$. Let G, G′ be the centres of the circles touching these and the ellipse, so that QG is normal to the ellipse, and GP′ goes through the contact-point T. Let the tangent at T meet the axes in α and β. Then

$A = 4\Delta CGP, \quad A' = 4\Delta C\alpha\beta.$

Let $CN = x.$

Now $r'^2 = QG^2 = b^2(1-e^2x^2/a^2),$

also $(r+r')^2 - r^2 = CG^2 = e^4x^2;$

hence $r'^2 + r'b = e^2(e^2x^2) = e^2(a^2 - a^2r'^2/b^2);$

$\therefore \quad r'^2(1 + a^2e^2/b^2) + r'b - a^2e^2 = 0; \quad \therefore \quad r'^2 + r'b(1-e^2) - b^2e^2 = 0;$

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therefore $(r'+b)(r'-be^2)=0.$

Therefore $r'=be^2=2re^2$, so that $r'/r=2e^2$.

This result may be obtained geometrically also. Hence

$$\text{cosec PGC}\,(=\theta)=(r+r')/r=2e^2+1, \quad \text{and} \quad \cot^2\theta=4e^4+4e^2.$$

Now $$\frac{A}{A'}=\frac{CG\,.\,CP}{C\alpha\,.\,C\beta}=\frac{CP^2\cot\theta}{C\beta^2\tan\theta}=\frac{r^2}{r'^2}\cot^2\theta=\frac{1}{4e^4}(4e^2+4e^4)=1+\frac{1}{e^2}.$$

13671. (R. F. Davis, M.A.)—If ABCD is a quadrilateral whose diagonals AC, BD intersect in O, and α, β, γ, δ are the circumcentres of BCD, ACD, ABD, ABC, respectively, prove that $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\alpha$ are inversely proportional to the perpendiculars from O on CD, DA, AB, BC, respectively.

Solution by F. H. Peachell, B.A.; *and* W. E. Jeffares, B.A.

Draw perpendiculars OL, OM on AD, AB.

Now, because α, β, γ, δ are centres, $\beta\gamma$ is perpendicular to AD, $\delta\beta$ perpendicular to AC, $\gamma\delta$ perpendicular to AB, &c.; therefore

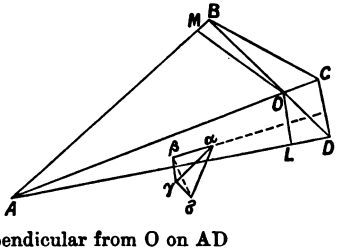
$$\angle\,\delta\beta\gamma=\text{CAD},$$

$$\angle\,\beta\delta\gamma=\text{BAC, \&c.};$$

$$\therefore \quad \frac{\delta\gamma}{\beta\gamma}=\frac{\sin\delta\beta\gamma}{\sin\beta\delta\gamma}$$

$$=\frac{\sin\text{CAD}}{\sin\text{BAC}}=\frac{\text{OL}}{\text{OM}}=\frac{\text{perpendicular from O on AD}}{\text{perpendicular from O on AB}}.$$

Similarly for rest.



13679. (H. W. Curjel, M.A.)—An elastic band passes round two fixed pegs and a pencil point. Find the curve described by the latter when the resultant pressure exerted by the band on it is constant.

Solution by Rev. J. Cullen, B.A.

Let a be the distance between the pegs, $k+a$ the original length of the band. Then at a tension T we have

$$(r+r'-k)/\text{T}=(k+a)/\text{E};$$

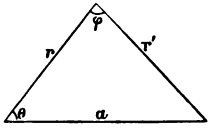
also $$\text{R}=2\text{T}\cos\tfrac{1}{2}\phi.$$

Therefore $$(r+r'-k)=c\sec\tfrac{1}{2}\phi,$$

where c is a constant,

$$\sin\phi=a\sin\theta/r', \quad \text{and} \quad r'^2=r^2+a^2-2ra\cos\theta;$$

whence the polar equation of the locus is obtained by eliminating ϕ.



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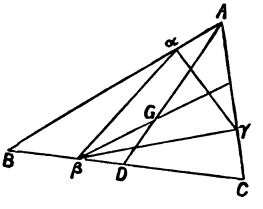
13713. (Rev. T. Mitchson, B.A.)—In the triangle ABC, take a point α in AB such that $A\alpha = \frac{1}{3}$ of AB, and, similarly, $B\beta = \frac{1}{3}$ of BC, and $C\gamma = \frac{1}{3}$ of CA. In the triangle $\alpha\beta\gamma$, prove, *inter alia*, (1) that each side is parallel to one median of ABC and $\frac{2}{3}$ its length, and is trisected by the other two; (2) that the centroid of both triangles is the same; and (3) that each median of $\alpha\beta\gamma$ is parallel to a side of ABC and half its length.

Solution by L. E. Reay, B.A.; W. E. Jeffares, B.A.; *and the* Proposer.

(1) $B\alpha = \frac{2}{3}AB$, $B\beta = \frac{1}{3}BC = \frac{1}{3}BD$; therefore $\alpha\beta$ is parallel to AD, and equals $\frac{1}{3}AD$. Also, drawing through γ a line parallel to AD or $\alpha\beta$, D divides in ratio 1 : 2 the intercept on BC between β and this line. Therefore AD divides $\beta\gamma$ and $\alpha\gamma$ in ratio 1 : 2. Therefore $\beta\gamma$, &c., are trisected by the medians AD, &c.



(2) Again, since lines through centroid of ABC parallel to sides $\alpha\beta$, &c., divide these sides in ratio 1 : 2, the centroid of ABC is also that of $\alpha\beta\gamma$.

(3) Also median of $\alpha\beta\gamma$ through β divides BD in ratio 2 : 1, and AD in the same ratio since it passes through G; therefore it is parallel to AB. Also from figure it equals $\frac{2}{3}AB - \frac{1}{3}AB = \frac{1}{3}AB$.

13608. (Professor H. Langhorne Orchard, M.A., B.Sc.)—In a circle of radius a is inscribed the maximum quadrilateral; in this quadrilateral is inscribed a circle, and in this circle the maximum quadrilateral; then in this quadrilateral another circle, and so on, *ad infinitum*. Find the ratio of the sum of the areas of all the quadrilaterals to the sum of the areas of all the circles.

Solution by H. W. Curjel, M.A.; *and many others.*

The maximum quadrilateral is a square. Therefore the ratio required = area of a square : area of its circumcircle = $1 : \pi/2 = 2 : \pi$.

13641. (W. J. Dobbs, M.A.)—The normal at P to a parabola (vertex A) meets the curve again at Q. The straight line drawn through A perpendicular to AQ meets the tangent at P in R. Prove that the locus of R is a straight line, and that QR is divided by the axis in the ratio 2 : 1.

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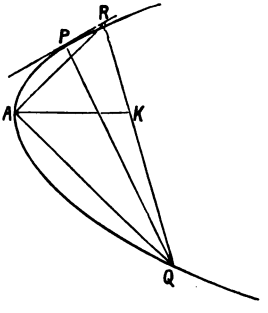
Solution by E. P. BARRETT, B.A.; Professor A. DROZ-FARNY; *and many others.*



Let RQ cut axis in K. Let P be $(at^2, 2at)$, and Q be $(at_1^2, 2at_1)$; then

$$t_1 = -(t^2+2)/t,$$

and ordinate of Q is

$$[-2a(t^2+2)]/t.$$

AQ is $2x = t_1 y$;

$\therefore$ AR is $2y + t_1 x = 0$.

Equation to tangent at P is

$$ty = x + at^2;$$

therefore R is point $[2a, a(t^2+2)/t]$,

whence locus of R is straight line $x = 2a$,

and K divides QR in ratio 2 : 1.

Note on the Factorization of Integers of form $10n + 2r - 1$.

By R. W. D. CHRISTIE.

For the further analysis of integers I propose the following:—

(1) *Numbers of the form* $10n + 7$:

$$N = (10a \pm 1)(10b \mp 3) = 100ab \pm 30a \mp 10b - 3;$$

therefore $(N+3)/10 = 10M = 10ab \pm (3a - b)$;

thus, $M = ab \pm (3a-b)/10$, to find a and b, M being known. Consequently

$$3a = (3a-b)/2 + \{[(3a-b)/2]^2 \pm [3(3a-b)]/10 + 3M\}^{\frac{1}{2}}.$$

$$b = -(3a-b)/2 + \{[(3a-b)/2]^2 \pm [3(3a-b)]/10 + 3M\}^{\frac{1}{2}}.$$

Using *all adventitious aids* composites below a million are easily analysed.

Rule.—Take the digits $(N+3)/100$, and equate to $ab \pm (3a-b)/10$.

E.g., 8357. Here $83{\cdot}6 = ab + (b - 3a)/10 = 6 \times 14 + (14 - 18)/10$. We know b is 14 and not a, since we get $-{\cdot}4$ to counterbalance ·6, for ·6 + ·4 = the additional unit.

The factors therefore $(10 \times 6 + 1)(10 \times 14 - 3) = (61)(137)$.

(2) *Numbers of the form* $10n + 3$:

$$N = (10a \pm 1)(10b \pm 3) = 100ab \pm 30a \pm 10b + 3;$$

$\therefore$ $(N-3)/10 = 10M = 10ab \pm (3a+b)$; thus $M = ab \pm (3a+b)/10$.

Consequently $3a = (3a+b)/2 + \{[(3a+b)/2]^2 \pm [3(3a+b)/10] - 3M\}^{\frac{1}{2}}$,

$$b = (3a+b)/2 - \{[(3a+b)/2]^2 \pm [3(3a+b)/10] - 3M\}^{\frac{1}{2}}.$$

E.g., 9313. Here $93{\cdot}1 = ab - (3a+b)/10 = 7 \times 14 - [(3 \times 14 + 7)/10]$. We *know* $a = 14$ because we thus get the decimal ·9 required to complete the unit.

The factors are $(10 \times 14 - 1)$ and $(10 \times 7 - 3)$.

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(3) *Numbers of the form* $10n+1$:

$N=(10a\pm1)(10b\pm1)$ or $(10a\pm3)(10b\mp3)=100ab\pm(10a+10b)+1$

or $100ab\pm30(a-b)-9$;

$\therefore$ $(N-1)/10$ and $(N+9)/10=10M=10ab\pm(a+b)$ or $=10ab\pm3(a-b)$.

Consequently $a=(a+b)/2+\{[(a+b)/2]^2\pm(a+b)/10-M\}^{\frac{1}{2}}$,

$b=(a+b)/2-\{[(a+b)/2]^2\pm(a+b)/10-M\}^{\frac{1}{2}}$,

or $a=-(a-b)/2+\{[(a-b)/2]^2-[3(a-b)]/10+M\}^{\frac{1}{2}}$,

$b=-(a-b)/2-\{[(a-b)/2]^2-[3(a-b)]/10+M\}^{\frac{1}{2}}$,

E.g., 22991. Here $230=ab\pm3(a-b)/10$. We know *a and b end in the same integer;* therefore 28×8 are the only integers possible.

The factors are $(10\times28-3)(10\times8+3)$.

(4) *Numbers of the form* $10n+9$:

$N=(10a\pm3)(10b\pm3)$ or $(10a\pm1)(10b\mp1)=100ab\pm30(a+b)+9$

or $100ab\pm10(a-b)-1$;

$\therefore$ $(N-9)/10$ or $(N+1)/10=10M=10ab\pm3(a+b)$ or $10b\pm(a-b)$;

thus $M=ab\pm3(a+b)/10$ or $ab\pm(a-b)/10$.

Consequently $a=(a+b)/2+\{[(a+b)/2]^2\pm[3(a+b)]/10-M\}^{\frac{1}{2}}$,

$b=(a+b)/2-\{[(a+b)/2]^2\pm[(3a+b)]/10-M\}^{\frac{1}{2}}$,

or $a=(a-b)/2+\{[(a-b)/2]^2-(a-b)/10+M\}^{\frac{1}{2}}$,

$b=-(a-b)/2+\{[(a-b)/2]^2-(a-b)/10+M\}^{\frac{1}{2}}$.

E.g., 91339. Here $913\cdot4=ab+(a-b)/10=38\times24+(38-24)/10$.

It will be found that these expressions for finding the roots can be simplified: *e.g.*, to analyse composites ending in the digit 7 by making $x=(3a-b)/2$, we get

$$3a=x+\{x^2+3x/5+3M\}^{\frac{1}{2}},\qquad b=x-\{x^2-3x/5+3M\}^{\frac{1}{2}},$$

and similar expressions for the others.

It is supposed that the original number N throughout is not of form 3M, and therefore for the factor

$10a+1$,	$a=3a_1$	or	$3a_1+1$ only,
$10b+1$,	$b=3b_1$	or	$3b_1+1$,,
$10a-1$,	$a=3a_1$	or	$3a_1-1$,,
$10b-1$,	$b=3b_1$	or	$3b_1-1$,,
$10b+3$,	$b=3b_1+1$	or	$3b_1-1$,,
$10a+3$,	$a=3a_1+1$	or	$3a_1-1$,,
$10b-3$,	$b=3b_1+1$	or	$3b_1-1$,,
$10a-3$,	$a=3a_1+1$	or	$3a_1-1$,,

With this understanding the method is perfectly general for all integers ending in 1, 3, 7, or 9.

To solve these equations, it is sometimes advantageous to employ the diophantine analysis: *e.g.*, solve

$$\{[(a+b)/2]+(a+b)/10-[ab+(a+b)/10]\}^{\frac{1}{2}};$$

i.e., assume $x^2+x/5-M=(x-b)^2$, where M is known.

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Then $x = [5(b^2 + M)/(10b+1)$ and $a = (10M - b)/(10b+1)$.

I append a few notes on some principles of factorization.

I. *Multiplication.*—If the digits on the left are a multiple of those on the right, or *vice versâ*, the factors may be written down at sight. *E.g.*, factorize 381127. Since $381 = 3 \times 127$, the factors are 3001 and 127. Similarly, $29197 = 7 \times 43 \times 97$. Again, $179537 = 59$ (M).

II. *Addition.*—Factorize 1800679. Here $679 + 18 = 697 = 41$ (M) and, the period of 41 being 5, therefore $1800679 = 41$ (M). Again

$$2360000003 = 239 \text{ (M)}.$$

An extension of the principle is as follows:—

Let P be any integer prime to 10, p the period of 1/P, n any integer, and d any of the ten digits. Take n groups of $p-1$ similar digits (d's), and let $nd = a + b + c +$ &c.; then $addd \ldots bddd \ldots eddd \ldots = 0 \bmod P$.

Ex.—Let $P = 37$, $n = 2$, $d = 5$; thus $nd = 10 = 1 + 9 = 2 + 8 =$ &c. Then 155955 or 255855 or $a55b55 \Rightarrow 0 \bmod 37$.

Consequently, to analyse 10397 or 17059, we simply reverse the process by multiplying these numbers by 15, and the *form* of the product thus found detects the factor 37.

III. *Square* (initial).—Let $x =$ tens' digit, $y =$ units' digit, $z = 100 - (10x + y)$, $b = za$; then

$$N = (10a+1)(10b - z) = (10a-1)(10b+z),$$

and $(N+z)/100 = za^2$, whence a, thence factors.

Factorize 46787. Here $468 = 13a^2$; therefore $a = 6$. Hence $46787 = 13 \times 61 \times 59$.

Extending the principle,

$$N = (za^3 - 1)t^3 + x.t^2 + y.t + v = t^3.za^2 - (t^3 - x.t^2 - y.t - v) = z(a^2t^3 - 1).$$

Factorize 2422319. First multiply by 21, giving 50868699, and note $50868 + 1 = 301 \times 13^2$: therefore 43 and 56333 are factors.

IV. *Square* (final).—Let $N = (a+s)(b+s) = ab + s(a+b) + s^2$.

Analyse 2312181. Assume the last two figures to come from a square, say $91^2 = 8281$; therefore $23121 - 82 = 23039$. And

$$23039 = 204 \quad = 12 \times 17,$$
$$2639 = 91\,(12 + 17).$$

The factors are 1291, 1791.

V. *Division.*—$N = (mat + x)(nat + x)$, where $x = 1$, 3, 7, or 9; $a =$ common factor of the numbers formed by all the digits except the units; $t = 10$. Therefore $N = at\{mnat + (m+n)x\} + 1$ or 9,

or $$at\{mnat + (m+n)x + 4 \text{ or } 8\} + 9 \text{ or } 1.$$

Thus a is found. *E.g.*, $N = 121771$. Here $12177 = 11 \times 1107$; therefore $a = 11$, and we have also $1107 = 110mn + m + n$, whence easily $m = 5$, $n = 2$, whence $ma = 55$, $na = 22$, and the factors 551, 221.

Again factorize 126439. Here $12643 - 4 = 12639 = 11 \times 1149$; thus $a = 11$ and $1149 = 110mn + 7(m+n)$, whence $m = 5$, $n = 2$, and ma, $na = 55$, 22, and the factors are 557, 227.

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VI. *Tens' Form.*—We can determine the *form of the factors* by a consideration of the form of the number itself, *supposing the units' digit omitted.* *E.g.*, if N ends in unity, and the number as supposed is of form $6\omega+3$, whether the factors turn out to be each of the form $10n\pm1$, we know that they must be of the form

$$\{(6r+s)\,t+1\}\,\{(6r_1+s+3)\,t+1\}$$

or

$$\{(6r+s)\,t+9\}\,\{(6r_1+s+3)\,t+9\};$$

i.e., the *tens* are of a constant form. But, should the factors be of the form $10n\pm3$, we, at any rate, know that the factors must be of the form

$$N=\{(6r+s)\,t+7\}\,\{(6r_1+s+1)\,t+3\}.$$

And it will be found that whenever the tens take a certain form so do the factors.

It is easy also to show that, if the tens' digit be *odd*, the tens' digit *of the factors* are of opposite affection and *vice versâ*.

E.g., analyse 712111. Here $71211=6\omega+3$. Consequently the factors take the form $(6r+1)\,t+9$ and $(6r_1+4)\,t+9$.

A table of the forms would be highly useful.

I append one such table for composites ending in 7.

(1) *Factors ending in* 1 *and* 7.

$$\begin{aligned}
N=\{(6r\pm s)\,t+1\}\,\{(6r_1+s)\,t-3\}, &\quad \text{Tens' form} = 6\omega+1,\\
\{(6r+s)\,t+1\}\,\{(6r_1-s)\,t-3\}, &\quad ,, = 6\omega+3,\\
\{(6r+s)\,t+1\}\,\{(6r_1+s+1)\,t-3\}, &\quad ,, = 6\omega,\\
\{(6r+s)\,t+1\}\,\{(6r_1+s+2)\,t-3\}, &\quad ,, = 6\omega+1,\\
\{(6r+s)\,t+1\}\,\{(6r_1+s+3)\,t-3\}, &\quad ,, = 6\omega+4,\\
\{(6r+s)\,t+1\}\,\{(6r_1+s+4)\,t-3\}, &\quad ,, = 6\omega+3,\\
\{(6r+s)\,t+1\}\,\{(6r_1+s+5)\,t-3\}, &\quad ,, = 6\omega+4.
\end{aligned}$$

Ex.—Let $r=17$, $s=3$, $r_1=19$; then $N=1051\times1177=1237027$, and we find that $123702=6\omega$, as it should do. In analysing a number, we observe the form of the tens, and thence the form of the factors. It must be carefully observed that none of these factors are of form 3M.

(2) *Factors ending in* 9 *and* 3.

$$\begin{aligned}
N=\{(6r\pm s)\,t-1\}\,\{(6r_1\pm s)\,t+3\}, &\quad \text{Tens' form} = 6\omega+1,\\
\{(6r+s)\,t-1\}\,\{(6r_1-s)\,t+3\}, &\quad ,, = 6\omega+3,\\
\{(6r+s)\,t-1\}\,\{(6r_1+s+1)\,t+3\}, &\quad ,, = 6\omega+4,\\
\{(6r+s)\,t-1\}\,\{(6r_1+s+2)\,t+3\}, &\quad ,, = 6\omega+3,\\
\{(6r+s)\,t-1\}\,\{(6r_1+s+3)\,t+3\}, &\quad ,, = 6\omega+4,\\
\{(6r+s)\,t-1\}\,\{(6r_1+s+4)\,t+3\}, &\quad ,, = 6\omega+1,\\
\{(6r+s)\,t-1\}\,\{(6r_1+s+5)\,t+3\}, &\quad ,, = 6\omega.
\end{aligned}$$

Ex.—Taking, as before, $r=17$, $s=3$, $r_1=19$, then

$$N=1049\times1223=1282927,$$

and we find that $128292=6\omega$, as it should do.

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Noting the fact that the forms $6\omega + 2$, $6\omega + 5$ are absent, we at once see that this must be so, since N is supposed not to equal 3M. For composites in 7 there is the unique form 6ω, and for integers in 3 a corresponding form $6\omega + 1$. It seems preferable (as there thus are *only two cases*) to bring integers in 9 and 1 either to 7 or 3 first, and then factorize. I append a plan of a Factorization Table, with an explanation of its mode of formation and use. The full table can be easily constructed (up to 100) in an hour. It may be extended indefinitely either vertically or horizontally.

Factorization Table.

Zeros	7	13	17	19	23	29	31	41	43	47	53	59	67	71	73	79
0	2	9	5	17	16	26	3	4	30	14	37	53	20	7	51	
1	3	10	9	15	20	20	22	25	3	39	9	23	2	22	27	
2	1	1	6	11	2	2	27	23	39	18	38	20	27	59	10	
3	5	4	4	3	14	6	12	31	34	30	25	2	63	13	1	
4	4	3	14	6	6	18	26	40	12	3	29	12	13	51	22	
5	6	12	15	12	19	25	15	4	27	5	40	13	8	69	46	

Mode of Formation.—(1) Integers in *unity*. Place the tens' digit at the head of the table. (2) Integers in *three*. Bring to unity and take the tens' digit. Thus $23 \times 7 = 16{\cdot}1$. (3) Similarly for integers in *seven* and *nine*. Thus $17 \times 3 = 5{\cdot}1$. The general formula is $(XP-1)/10$. To get the remaining numbers I proceed as follows, taking 7 as a sample. Subtracting the two top numbers, say 2 from 7, I get 5, and use it as a multiplier. Now say 5×2 and cast out the prime wherever necessary.

Use.—Factorize 2900001. Running the eye along the four zero line, we detect 53 as a factor, and, since $29 = 13m + 3$ also, 13 is another. Tables for 3, 7, and 9 are formed similarly. Is 100000000000003 a prime? The table before me answers No, for 19 is a factor.

Or we may form tables for any specified numbers prime to 10; *e.g.*, 19. The multiplier for 19 is 2. For primes in 9 add 1 to the tens' digit, and you have the multiplier.

Zeros									
0	1·9	3·8	5·7	7·6	9·5	11·4	13·3	15·2	17·1
1	2·09	6·08	10·07	14·06	18·05	3·04	7·03	11·02	15·01
2	4·009	12.008	1·007	9·006	17·005	6·004	14·003	3·002	11·001
&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.

Is 1000000000000000000007 a prime? No. The table shows that $1{\cdot}007 = 19$ (M), and therefore $10^{21} + 7$, or $10^{18m+3} + 7 = 19$ (M).

Here note 18 is the *period* of 1/19.

From these tables we can obtain a contribution to the solution of the unconquered problem in the shape of two sets of theorems: one obtained horizontally and the other vertically.

Space forbids the whole of the formulæ, but from what is given the rest can readily be got.

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Numbers of Form $10N+1$.

(1) ..... $(P-1)/10\left\{(9P+1)/10\right\}^{n-1} = (P-1)(10^{p-n}) = q \bmod P$; here P ends in unity.

(2) ..... $(7P-1)/10\left\{(3P+1)/10\right\}^{n-1} = (P-1)(10^{p-n}) = q \bmod P$; here P ends in 3.

(3) ..... $(3P-1)/10\left\{(7P+1)/10\right\}^{n-1} = (P-1)(10^{p-n}) = q \bmod P$; here P ends in 7.

(4) ..... $(9P-1)/10\left\{(P+1)/10\right\}^{n-1} = (P-1)(10^{p-n}) = q \bmod P$; here P ends in 9.

Ex.: $P = 53$, $p = 13$ = period of 1/53, $n = 5$, say.

Then $37.16^4 = 52.10^8 = 29 \bmod 53$ and $2900001 = 53$ (M);

or let $P = 41$, $p = 5$, $n = 3$; then $4.37^2 = 40.10^2 = 23 \bmod 41$, and $\cdot 23001 = 41$ (M).

For numbers of form $10N+3$, and when P also ends in 3, we have the formula

(5) ..... $(P-3)/10\left\{(3P+1)/10\right\}^{n-1} = (P-3)(10^{p-n}) = q \bmod P$,

and generally

(6) ..... $10^{p-n} = \left\{XP+1)/10\right\}^n \bmod P$, for primes ending in X.

Let $P = 41$, $p = 5$, $n = 3$; then $10^2 = 37^3 \bmod 41 = 18 \bmod 41$.

5451. (T. Cotterill, M.A.) — In the resultant of a quadratic $Ax^2+Bxy+Cy^2$, and the general form $\phi(x, y)$, the part independent of B is $\phi(x, y) \cdot \phi(x, -y)$, in which $x^2 = +C$, and $y^2 = -A$, and consequently the sum or difference of two real squares according as A and C have or have not the same sign. The parts independent of A and C are, respectively, $\phi(1, 0) \cdot \phi(C, -B)$ and $\phi(0, 1) \cdot \phi(-B, A)$.

Solution by Rev. J. Cullen, B.A.

Let $\phi(x, y) = 0$ and $\psi(x, y)$ be two homogeneous equations of the mth and nth degrees. Then

$$R = (-1)^{mn} b_n^m \phi(1, \beta_1) \phi(1, \beta_2) \dots \phi(1, \beta_n)$$

where b_n is the coefficient o f y in ψ and $\beta_1, \beta_2, \dots \beta_n$ the roots of $\psi(1, u) = 0$. Hence, writing $Ax^2+Bxy+Cy^2$ for $\psi(x, y)$, we find the part of R independent of B by putting $B = 0$, so that

$$\beta_1 = +(-A/C)^{\frac{1}{2}} \quad \beta_2 = -(-A/C)^{\frac{1}{2}}.$$

Therefore the part of R independent of B is $\phi(x, y) \phi(x, -y)$ if $x^2 = +C$, $y^2 = -A$.

Putting $A = 0$, we have $\beta_1 = 0$, $\beta_2 = -B/C$; therefore part of R independent of A is $\phi(1, 0) \phi(C, -B)$; similarly, part independent of C is $\phi(0, 1) \phi(-B, A)$.

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13784. (W. J. C. MILLER, B.A.)—Find the value of the definite integral $\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\theta} \cos^4(\theta+\phi)\sec^2\phi\, d\theta\, d\phi$.

Solutions (1) *by* JOHN F. WILKINSON, B.A.; (2) *by* H. W. CURJEL, M.A.

(1) Expanding, by the binomial theorem, the developed form of $\cos(\theta+\phi)$, and multiplying each term by $\sec^2\phi$, the following five expressions result:—

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\theta}\Big[\cos^4\theta\cos^2\phi - 4\cos^3\theta\sin\theta\sin\phi\cos\phi + 6\cos^2\theta\sin^2\theta\sin^2\phi - 4\cos\theta\sin^3\theta\frac{\sin^3\phi}{\cos\phi} + \sin^4\theta\frac{\sin^4\phi}{\cos^2\phi}\Big]d\theta\, d\phi.$$

The integrations are easy but long.

The integral $\int\frac{\sin^3\phi}{\cos\phi}d\phi$ can be got by putting $z=\cos\phi$, and $\int\frac{\sin^4\phi}{\cos^2\phi}d\phi$ by integrating by parts.

On inserting the values of the limits of ϕ, we get a long series of expressions in θ, each of which is easy to integrate. The final result $=\frac{1}{4}$.

(2) The given integral clearly

$$= \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\phi}\cos^4(\theta+\phi)\sec^2\phi\, d\phi\, d\theta$$

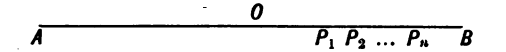
$$= \int_0^{\frac{1}{2}\pi}\Big[\frac{3\pi}{16}\sec^2\phi - \tfrac{3}{8}\phi\sec^2\phi - \tfrac{3}{8}\tan\phi - \frac{\sin\phi\cos\phi}{4}\Big]d\phi$$

$$= {}_0^{\frac{1}{2}\pi}\Big[\frac{3\pi}{16}\tan\phi - \frac{3\phi}{8}\tan\phi + \frac{\cos 2\phi}{16}\Big] = \tfrac{1}{4}.$$

13763. (Rev. W. ALLEN WHITWORTH, M.A.)—There are n equal sugar-sticks. Each stick is broken into two pieces, all positions of the fracture being equally likely. Of the $2n$ pieces thus formed a child is to take the largest. Show that his expectation is $\frac{2n+1}{2(n+1)}$ of a stick.

Solution by the PROPOSER.

Let AB be the length of a stick. Bisect AB in O.



Let each stick be laid along AB with its larger part towards A. Let $P_1, P_2, \ldots P_n$ be the points of fracture, P_n being the furthest from A. Then AP_n is the length which the child will take.

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Now P_1, P_2, ... P_n are random points on OB. Therefore, by a known theorem (*DCC Exercises*, 497), they divide OB into $n+1$ segments of equal expectation. Hence

$$E(P_nB) = OB/(n+1) = AB/2(n+1);$$

and therefore $$E(AP_n) = AB(2n+1)/2(n+1).$$

[Mr. CURJEL solves the Question as follows:—Let a = the length of each stick. Then expectation

$$= \left(\int_0^{\frac{1}{2}a} (\tfrac{1}{2}a + x)\, x^{n-1}\, dx\right) \Big/ \left(\int_0^{\frac{1}{2}a} x^{n-1}\, dx\right)$$

$$= \frac{\frac{1}{2}a\left\{1/n + 1/(n+1)\right\}}{1/n} = \frac{2n+1}{2(n+1)}\, a.]$$

13776. (W. C. STANHAM, B.A.)—If three straight lines, one through each of the angular points of an equilateral triangle, making angles ψ_1, ψ_2, ψ_3 with the corresponding opposite sides are concurrent,

$$\cos(\psi_2+\psi_3-\psi_1) + \cos(\psi_3+\psi_1-\psi_2) + \cos(\psi_1+\psi_2-\psi_3) = 0,$$

and, conversely, if this relation exists, the lines are concurrent. The following convention is adopted:—Let D, E, F be the points at which the lines meet the sides BC, CA, AB of ABC. Suppose AD, BE, CF rotated through small angles in the same one direction round A, B, C respectively, and that their new directions meet BC, CA, AB respectively at D′, E′, F′. Then $ADD' = \psi_1$, $BEE' = \psi_2$, $CFF' = \psi_3$.

Solution by W. H. SALMON, B.A.; F. H. PEACHELL, B.A.; *and the* PROPOSER.

(1) If AD, BE, CF are concurrent, to prove the relation

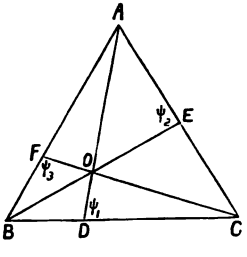
$$\cos(\psi_2+\psi_3-\psi_1) + \cos(\psi_3+\psi_1-\psi_2) + \cos(\psi_1+\psi_2-\psi_3) = 0.$$

Now $$\frac{AE \cdot BF \cdot CD}{CE \cdot AF \cdot BD} = 1.$$

But $$\frac{AE}{CE} = \frac{AE}{BE} \cdot \frac{BE}{EC}$$

$$= \frac{\sin(120^\circ - \psi_2)}{\sin 60^\circ} \cdot \frac{\sin 60^\circ}{\sin(\psi_2 - 60^\circ)}$$

$$= \frac{\sin(\psi_2 + 60^\circ)}{\sin(\psi_2 - 60^\circ)};$$



therefore $$\sin(\psi_1+60^\circ)\sin(\psi_2+60^\circ)\sin(\psi_3+60^\circ) = \sin(\psi_1-60^\circ)\sin(\psi_2-60^\circ)\sin(\psi_3-60^\circ) \quad \text{... (i.).}$$

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But, if α, β, γ be any three angles,

$$\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)=4\sin\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\gamma+\alpha)\sin\tfrac{1}{2}(\alpha+\beta).$$

Put $\beta+\gamma=2x$, $\gamma+\alpha=2y$, $\alpha+\beta=2z$;

therefore $\Sigma\sin(-x+y+z)-\sin(x+y+z)=4\sin x\sin y\sin z$;

therefore, from (i.),

$$\Sigma\sin(60^\circ+\psi_2+\psi_3-\psi_1)+\sin(\psi_1+\psi_2+\psi_3)$$
$$=\Sigma\sin(-60^\circ+\psi_2+\psi_3-\psi_1)+\sin(\psi_1+\psi_2+\psi_3);$$

therefore $\cos(\psi_2+\psi_3-\psi_1)+\cos(\psi_3+\psi_1-\psi_2)+\cos(\psi_1+\psi_2-\psi_3)=0.$

(2) Conversely, if the relation holds, AD, BE, CF are concurrent.

For, if they are not, let AO meet BC in D′, making $AD'C=\psi_1'$.

Then (ψ_2, ψ_3 being fixed) ψ_1, ψ_1' both satisfy the relation

$$\cos(\psi_2+\psi_3-\psi_1)+\cos(\psi_3+\psi_1-\psi_2)+\cos(\psi_1+\psi_2-\psi_3)=0,$$

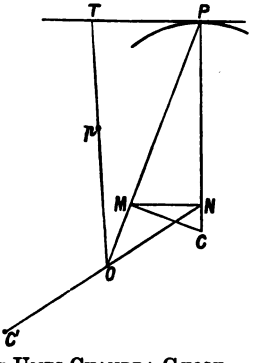
which is of the form $A\cos\psi_1+B\sin\psi_1=0$;

$\tan\psi_1=$ const., or $\psi_1=\psi_1'$; since both are $<\pi$.

13786. (B. H. Steede, M.A.)—Let P be any point on a given curve, and C be the corresponding centre of curvature. From C draw CM perpendicular to radius vector OP, and from foot of this perpendicular M draw MN perpendicular to the normal PC. Join N with the origin O, and produce NO, so that

$$NO : OC' = PN : k^2/p,$$

k being a constant, and p being the perpendicular from origin on tangent at P. Prove that the locus of C′ is the evolute of the reciprocal curve with respect to a circle whose centre is O and radius k.



Solution by H. W. Curjel, M.A.; Professor Umes Chandra Ghosh; *and others.*

Let Q be the point of the reciprocal curve corresponding to the tangent at P, and C_1 the centre of curvature at Q.

Let $OP=r$, $OT=p$, $PC=\rho$, $OQ=r'$ $QC_1=\rho'$, and $p'=$ perpendicular from O on tangent at Q.

Then $PN=\rho p^2/r^2$, $r'=k^2/p$, $p'=k^2/r$,

and $$\rho'/r'=dr'/dp'=\frac{r^2}{p^2}\frac{dr}{dp}=r\Big/\frac{p^2\rho}{r^2}=OP/PN.$$

But QC_1 is clearly parallel to OP. Therefore C_1 is on NO,

and $$NO : OC_1 = PN : OQ.$$

Therefore C_1, C′ coincide; hence the required result.

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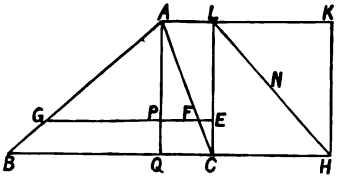
13806. (I. Arnold.)—Bisect a given triangle by a line parallel to the base by the first book of Euclid only.

Solution by the Proposer.

Let ABC be the triangle. Draw AK parallel to BC and complete the square CHKL. Join LH and bisect it in N. Make LE = LN, and draw EFPG parallel to BC. GF bisects the triangle ABC.



From analysis $BC^2 = 2GF^2$, and $AQ^2 = 2AP^2$;

therefore $AP = \frac{AQ}{\sqrt{2}}$, and $GF = \frac{BC}{\sqrt{2}}$;

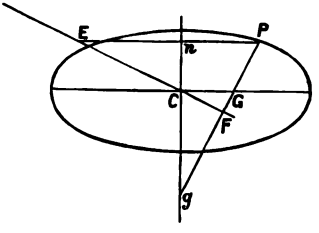
therefore $GF \cdot AP = \frac{1}{2}(AQ \cdot BC)$; and $\frac{1}{2}(GF \cdot AP = \frac{1}{4}AQ \cdot BC$,

or $\triangle AGF = \frac{1}{2}ABC$.

13595. (J. Macleod, M.A.)—From a point P on an ellipse whose centre is C a line PE parallel to the major axis meets the diameter of the ellipse parallel to the tangent at P in E; the same diameter meets the normal at P in F. If G be the intersection of the normal and minor axis and n be the intersection of PE and minor axis, show that the tangent from P to the circle circumscribing ECn is constant, and that the tangent from n to the circle circumscribing CGF is equal to the ordinate of the auxiliary circle passing through P. Also, if the chord of contact JLS of n with respect to the circle CGF intersect CG in L, show that JL, CL are to each other in the ratio of the major to the minor axis of the ellipse.

Solution by Lionel E. Reay, B.A.; Rev. J. Cullen, B.A.; *and the* Proposer.

(Note, in accordance with the more usual notation, G and g are the intersections with the major and minor axis respectively.)



(1) Square on tangent from P to circle CnE = PE . Pn
= PF . Pg = AB^2,
by known theorem.

(2) If P is (x', y'), and ellipse is
$x^2/a^2 + y^2/b^2 = 1$,
Pg is line $(x-x')/(x'/a^2) = (y-y')/(y'/b^2)$; $\therefore$ length of C$g = y'(a^2/b^2 - 1)$.
$\therefore$ $ng = y' \cdot a^2/b^2$; $\therefore$ $nC \cdot ng = PN^2 \cdot AC^2/CB^2$; $\therefore$ &c.

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(3) Circle CgF is $x^2+y^2+yCg=0$. Polar of n is $yy'+\frac{1}{2}Cg(y+y')=0$; therefore, substituting value of Cg, $CL=-y'(a^2-b^2)/(a^2+b^2)$, and

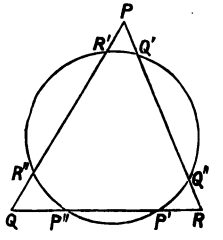
$$JL^2=-y'^2\left(\frac{a^2-b^2}{a^2+b^2}\right)^2+y'^2\frac{(a^2-b^2)^2}{(a^2+b^2)b^2}=y'^2\left(\frac{a^2-b^2}{a^2+b^2}\right)^2\frac{a^2}{b^2}=CL^2\frac{a^2}{b^2};$$

therefore $JL/CL=a/b$ numerically.

13705. (H. W. Curjel, M.A.)—Three conics α, β, γ are drawn through D, E, F, and β, γ cut again in A; γ, α in B; and α, β in C. A fourth conic, drawn through two of the points D, E, F, cuts α in G and H, β in K and L, γ in M and N. Show that, if the conics ADEFG, BDEFK, CDEFM meet in a fourth point, then the conics ADEFH, BDEFL, CDEFN also meet in a fourth point.

Solution by W. C. Stanham, B.A.

Let δ denote the fourth conic in the question, and let it pass through E and F. Project E and F into the circular points at infinity; then all the conics are projected into circles. With the projection of D as centre of inversion, invert the new figure; then we have to show that if a circle (the inverse of the projection of δ) cuts the sides of a triangle PQR in P', P'', Q', Q'', R', R'', respectively, and if PP', QQ', RR' are concurrent, then will PP'', QQ'', RR'' be concurrent. We have given therefore



$$RQ'.QP'.PR'=QR'.RP'.PQ',$$

and have to prove that $RQ''.QP''.PR''=QR''.RP''.PQ''$.

This is clearly the case, since

$$RP'.RP''=RQ'.RQ'',\ QP'.QP''=QR'.QR'',\ PQ'.PQ''=PR'.PR''.$$

In the figure, the sides of PQR represent (*i.e.*, are the inverses of the projections of) α, β, γ. P, Q, R represent A, B, C. The two triads of lines PP', QQ', RR' and PP'', QQ'', RR'' represent the two triads of conics in the question.

13719. (Rev. W. Allen Whitworth, M.A.)—A rod 9 feet long is to be divided into three parts, of which A. is to have the largest, B. the next, and C. the smallest. If the two fractures are made at random, A.'s, B.'s, and C.'s expectations will be respectively 66, 30, and 12 inches. But, if one fracture be made at random, and the larger portion of the rod be then divided at random, their expectations will be 64, 31, and 13 inches.

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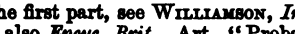
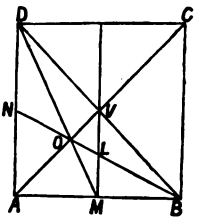
Solution by Professor CROFTON, F.R.S.

For the first part, see WILLIAMSON, *Int. Calc.*, p. 351; also *Encyc. Brit.*, Art. "Probability," p. 779. For the second part, let AB = a be the rod





let the first fracture be at X, which we may take always in the half MB of the rod, so that AX is always the largest portion; let the second fracture be at Y. Put AY = y, BX = x. We will take the number of random points in any length of the line, as measured by that length.

The whole number of ways N in which the rod may be divided will be $\iint dx\,dy$ extended to all values of x, y which satisfy $0 < x < \frac{1}{2}a$, $x + y < a$, $0 < y$; that is, if we draw a square on AB, and take x, y as coordinates of a point, A being the origin, (x, y) is confined to the figure AMVD; therefore

$$N = \tfrac{1}{8}a^2.$$

I. To find the mean value of the *least* part.

(1) When BX = x is the least part. The sum of all its values is $\iint x\,dx\,dy$, where $x < y$, $x < a - x - y$.

This restricts the point (x, y) to the triangle DOA, and the sum of all the values of x = DOA × abscissa of its C.G. = $a \cdot \frac{1}{6}a \cdot \frac{1}{3}a = \frac{a^3}{6 \cdot 9}$.

(2) When AY = y is the least part. The sum of its values is $\iint y\,dx\,dy$, where $y < x$, $y < a - x - y$, $x < \frac{1}{2}a$.

The point is confined to the figure OLMA, and the sum of values of y will be the areas of the triangles AOM, OLM multiplied respectively by the ordinates of their C.G., *i.e.*,

$$\tfrac{1}{2}a \cdot \tfrac{1}{3}a \cdot \tfrac{1}{3}a + \tfrac{1}{4}a \cdot \tfrac{1}{12}a \cdot \tfrac{1}{3}\left(\tfrac{1}{2}a + \tfrac{1}{4}a\right) = \frac{23}{16 \cdot 108}a^3.$$

The cases when YX is the least are the same as when AY is; hence the sum of all values of the least part is

$$2\frac{23}{16 \cdot 108}a^3 + \frac{a^3}{6 \cdot 9} = \frac{13}{32 \cdot 9}a^3.$$

Divide this by $\frac{1}{8}a^2$, the whole number of cases;

$$\text{Mean value of least part} = \tfrac{13}{108}a.$$

II. (1) When x is the *greatest* part. In $\iint x\,dx\,dy$, we have $x > y$, $2x + y > a$, $x < \frac{1}{2}a$; hence (x, y) is in OVM, and the sum of values of x is $\frac{a^3}{6 \cdot 9}$.

(2) When y is the greatest. Here we shall find the point is confined to the figure DNOV. The sum of the values of y will be found to be

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$\frac{1}{108}a^3$. Doubling this, for the cases when YX is greatest, and adding the above value for x, we find

$$\text{Mean value of greatest} = \left(\frac{11}{6.9}+\frac{1}{6.9}\right)a^3/\tfrac{2}{3}a^2 = \tfrac{6}{108}a.$$

For the intermediate part the mean value is of course $\frac{3}{108}a$.

[The PROPOSER'S solution does not involve the integral calculus.]

13756. (D. BIDDLE.)—Let N be a large composite odd number, whose factors are two or more primes unknown; and let $N^{\frac{1}{2}} = S + s$, where S is the integer and s is the decimal. Prove that, when $(S+k)^2 - N = \square$, N can be split up into two factors, of which $S+k$ = half the sum; $\square$ = the square of half their difference, whilst k, an integer determined by trial, = half the difference of their differences from S. Show also to what extent the value of k depends upon that of s.

[This method enables us to split up 8616460799 by taking $k = 56$ (which, owing to the nature of N in this instance, requires only 10 trials). In *Nature*, February 28, 1889, Dr. STANLEY JEVONS is quoted as saying that he was quite sure this number could not be factorized, as it was the product of two large primes. He was undoubtedly wrong, as indeed the writer in *Nature* (Mr. CHARLES J. BUSK, of Blackheath) has shown, and that by using a method to which the above, though discovered independently, bears a strong resemblance.]

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Let X, Y be the sought factors of N, so that $N = XY$; and suppose it has been found (by trial) that $N = (S+k)^2 - B^2$. Then

$$S+k+B = X, \quad S+k-B = Y;$$

therefore

$$\tfrac{1}{2}(X+Y) = S+k, \quad \tfrac{1}{2}(X-Y) = B;$$

$$k = \tfrac{1}{2}(X+Y) - S = \tfrac{1}{2}(X-S) - \tfrac{1}{2}(S-Y).$$

This proves all that was asked, except the dependence of k on s (where $s = \sqrt{N} - S$ = decimal part of $\sqrt{N}$).

Remarks.—This method of factorizing a number by separating it into a difference of squares, $N = (S+k)^2 - B^2$, was detailed at some length in a paper in *Nature*, Vol. XXXIX., p. 413, by Mr. C. BUSK, who gave therein a convenient method of computing the differences $\{(S+k)^2 - N\}$ for *successive* values of k. But, unless some guide be given for determining k, the number of trials required will be $= k$, so that this method (without some such clue to k) is far too laborious for factorizing large numbers ($N > 9$ millions), except when k is *quite small*, which happens when (and only when) the two factors X, Y are *nearly equal*. The present writer gave (*Messenger of Mathematics*, Vol. XX., p. 37) a method of choosing values of $(S+k)$, B, so as to make the last four figures of $(S+k)^2 - B^2$ agree

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with the last four figures of N. This greatly reduces the number of trials (which would otherwise be k) required; but, even with this help, the labour required when k is large would be simply prohibitive. But Mr. F. W. LAWRENCE has, in a series of recent papers (*Messenger of Mathematics*, Vol. XXIV., p. 100; *Quarterly Journal of Pure and Applied Mathematics*, No. CXI., 1896, p. 285; *Proceedings of the London Mathematical Society*, Vol. XXVIII., p. 465), effected *great* improvements in this method, by showing how to so greatly limit the possible forms of the larger square $(S+k)^2$ as to make this method *of practical use* for numbers of 10 or 11 figures, even when k is very large.

Editorial Note on Question 13756.

In order to show how the Dichotomous Factorization of N can be carried out more expeditiously than Col. CUNNINGHAM supposes, let N be of form $6n \pm 1$, which is the only form in which a composite number simulates a prime. Then $n = 6pq \pm (p \pm q)$, where p and q have values determined by $N = (6p \pm 1)(6q \pm 1)$, the factors of N being necessarily of similar form to itself (see Solutions of 13728, Vol. LXIX.). Now, since $6pq$ is always even, it is clear that $(p \pm q)$ must be odd or even with n. When n is odd, either p or q is odd and the other even; when n is even, p and q are both odd or both even, and their sum and also their difference must agree in this respect with n. In the formula $(S+k)^2 - N = \square$, $S+k$ represents half the sum of the factors of N, namely, $(6p \pm 1) + (6q \pm 1)$, and, where the signs differ, the half-sum $= 3(p+q)$. Therefore, in this case, if n be even, $S+k$ is a multiple of 6, because $p+q$ is even also; and if S/6 leave a remainder, $= r$, we can begin our trial series with $k = 6-r$, and go on to $12-r$, $18-r$, &c. When, on the contrary, the signs are similar, and n is even, then half the difference of the factors, of which the square is represented above by $\square$, will be $3(p-q)$, and this also will be a multiple of 6. When n is odd and the signs differ we can proceed by adding 3 instead of 6 to the last trial number k. When n is odd and the signs are similar we can proceed by adding 2 to k, since $3(p \pm q) \pm 1$ will then be even. Now the signs are similar when $N = 6n+1$; they differ when $N = 6n-1$.

But this is not our only guide. N is the difference between two square numbers, the endings of which are governed by its own to a wonderful degree. The final digits of square numbers are 0, 1, 4, 5, 6, 9; and there are twenty-two bidigital endings, namely,

00; 01, 21, 41, 61, 81; 04, 24, 44, 64, 84;
25; 16, 36, 56, 76, 96; 09, 29, 49, 69, 89.

Let the bidigital ending of N be added to each of these twenty-two in succession. In 60 per cent. of cases, provided $N = 6n \pm 1$, only two of the results will be also square endings. In the other 40 per cent. there will be five such results; never more.* These results are the trial-endings of the larger of the two squares, those of the smaller being the original square-ending in each sum.

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* The above percentages are arrived at by supposing the two factors of N to terminate at randon in 1, 3, 7, or 9. The products of pairs of these, where the factors may have the same final digit, terminate in 1 or 9 half as often again as in 3 or 7 (which alone are concerned in the more numerous square-endings).

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The final digit of numbers of form $6n \pm 1$ is 1, 3, 5, 7, or 9, of which 5 may be excluded. It is therefore easy to make a table giving the possible bidigital endings of the squares sought, for every bidigital ending of N that comes under our notice.

N.	*Squares.*	*N.*	*Squares.*	*N.*	*Squares.*	*N.*	*Squares.*
..01	00, 01; 24, 25	..03	01, 04; *	..07	09, 16; †	..09	00, 09; 16, 25
..11	25, 36; 89, 00	..13	16, 29; **	..17	04, 21; ‡	..19	81, 00; 25, 44
..21	00, 21; 04, 25	..23	01, 24; *	..27	09, 36; †	..29	00, 29; 96, 25
..31	25, 56; 69, 00	..33	16, 49; **	..37	04, 41; ‡	..39	61, 00; 25, 64
..41	00, 41; 84, 25	..43	01, 44; *	..47	09, 56; †	..49	00, 49; 76, 25
..51	25, 76; 49, 00	..53	16, 69; **	..57	04, 61; ‡	..59	41, 00; 25, 84
..61	00, 61; 64, 25	..63	01, 64; *	..67	09, 76; †	..69	00, 69; 56, 25
..71	25, 96; 29, 00	..73	16, 89; **	..77	04, 81; ‡	..79	21, 00; 25, 04
..81	00, 81; 44, 25	..83	01, 84; *	..87	09, 96; †	..89	00, 89; 36, 25
..91	25, 16; 09, 00	..93	16, 09; **	..97	04, 01; ‡	..99	01, 00; 25, 24

Those marked for reference have five pairs each, of which one only is given in the table, but which are given fully below :—

*	01	21	41	61	81	disposed with	04	24	44	64	84,
**	16	36	56	76	96	,, ,,	09	29	49	69	89,
†	09	29	49	69	89	,, ,,	16	36	56	76	96,
‡	04	24	44	64	84	,, ,,	01	21	41	61	81,

the difference between that on the left and that on the right being the bidigital ending of N.

N.B.—In all cases the second of each pair represents the trial ending of the larger square.

The following is a table of root-endings for each bidigital square-ending :—

01 ... 01, 49, 51, 99	04 ... 02, 48, 52, 98
21 ... 11, 39, 61, 89	24 ... 32, 18, 82, 68
41 ... 21, 29, 71, 79	44 ... 62, 88, 12, 38
61 ... 31, 19, 81, 69	64 ... 92, 58, 42, 08
81 ... 41, 09, 91, 59	84 ... 22, 28, 72, 78
16 ... 04, 46, 54, 96	09 ... 03, 47, 53, 97
36 ... 44, 06, 94, 56	29 ... 23, 27, 73, 77
56 ... 84, 66, 34, 16	49 ... 43, 07, 93, 57
76 ... 24, 26, 74, 76	69 ... 63, 87, 13, 37
96 ... 64, 86, 14, 36	89 ... 83, 67, 33, 17

00 ... 00, 10, 20, 30, 40, 50, 60, 70, 80, 90
25 ... 05, 15, 25, 35, 45, 55, 65, 75, 85, 95.

Now, it will be easy to see that the root-ending of the larger square in this second trial-scheme must coincide with the trial-value of $S + k$ in the first scheme; and this immensely reduces the number of trials.

Thus, the number referred to in the Question, $8616460799 = 6n - 1$, and n is even. Therefore $S + k = M(6)$; $S = 92824$. Therefore the trial-values of $S + k$, by the first scheme, have the successive bidigital endings 26, 32, 38, 44, 50, 56, 62, 68, 74, 80, and the last proves successful, being the tenth. This scheme alone, therefore, reduces the trials from 56 ($= k$) to 10. But now let us combine the two schemes. The above

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tables show that when N ends with ..99, the larger of the two squares of which N is the difference ends either with ..00 or ..24; and that the square root of the larger square accordingly may end in 0, 18, 32, 68, 82. Now, on comparing these with the trial endings of the first scheme, we find that the two schemes coincide in giving as trial-endings 32, 50, 68, 80. Therefore, by combining the two schemes, *four trials alone* are needed to factorize the number which Dr. JEVONS pronounced impracticable. By using *tridigital* endings the trials in the given instance are reduced to *two*; for 868 and 880 are the only root-endings (within the specified range) in which the two systems then coincide.

13769. (J. WOODCOTE WEST.)—N is a number of the form $a(a+1)/2$ of the form $3\beta^2-3\beta+1$; then $N = \square$ is impossible.

Solution by R. W. D. CHRISTIE.

We have $$T_a = 6T_{\beta-1}+1 = y^2.$$

Now T_a can end in the digits 0, 1, 3, 5, 6, 8 only; therefore $6T_{\beta-1}+1$ can end in the digits 1, 7, 9 only. Thus $T_a = 6T_{\beta-1}+1$ must end in unity $= 10m+1$. But $6T_a+1 = 60m+7 = \square$, which is clearly impossible, since no square can end in 7.

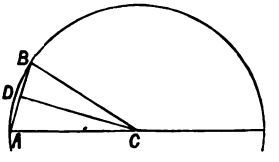
13734. (The late T. S. DAVIES.)—If the radius of a circle be divided in extreme and mean ratio, the greater segment is the side of a regular decagon inscribed in that circle, and the sum of the squares of the radius and its greater segment is equal to the square of the side of the inscribed regular pentagon.

Solution by LIONEL E. REAY, B.A.; *and* I. ARNOLD.

If AB, equal to the greater segment of the radius AC, is inscribed in the circle, then, by Euc. IV. 10,

$$\angle BAC = 2\angle ACB;$$

therefore $\angle ACB$ is one-tenth of four right angles; therefore AB is side of regular decagon inscribed in the circle.



Also, since $\sin 36° = \frac{1}{4}(\sqrt{10-2\sqrt{5}})$, the square on the side of the regular pentagon inscribed in the circle is

$$\tfrac{1}{4}(10-2\sqrt{5})\,AC^2 = AC^2+\tfrac{1}{4}(6-2\sqrt{5})\,AC^2 = AC^2+\{\tfrac{1}{2}(-1+\sqrt{5})\}^2 AC^2$$
$$= AC^2+AB^2.$$

13644. (W. E. JEFFARES, B.A.) — P is a point on the surface of the sphere from which tangent arcs are drawn to two equal non intersecting small circles, such that the product of the tangents of halves of these

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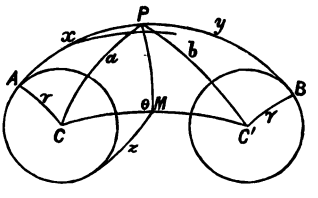
arcs is equal to the square of the tangent of the arc drawn from the point M midway between the centres C and C′ of the two circles. Let β be the arc MP, and θ the angle between MP and CC′. Prove that

$$\tan^2 \tfrac{1}{2}\beta = A \cos^2\theta + B \sin^2\theta,$$

where A and B are functions of the radius of the circles and of the distance between the centres which it is required to determine.

Solution by the PROPOSER.

Let PA $= x$, PB $= y$; let r be the radius of each of the circles, PM $= \beta$, and CC′ the arc joining the centres of the circles PC $= a$ and PC′ $= b$, and CC′ $= 2d$ and $\angle$ CMP $= \theta$.



$$\tan^2 \tfrac{1}{2}x = \frac{1-\cos x}{1+\cos x} = \frac{1-(\cos a/\cos r)}{1+(\cos a/\cos r)} = \frac{\cos r - \cos a}{\cos r + \cos a}$$

(since PCA is a spherical right-angled triangle).

Similarly, $\tan^2 \tfrac{1}{2}y = \dfrac{\cos r - \cos b}{\cos r + \cos b}$ and $\tan^2 \tfrac{1}{2}z = \dfrac{\cos r - \cos d}{\cos r + \cos d}$;

$\tan \tfrac{1}{2}x \tan \tfrac{1}{2}y = \tan^2 \tfrac{1}{2}z$ (by hypothesis); $\tan^2 \tfrac{1}{2}x \tan^2 \tfrac{1}{2}y = \tan^4 \tfrac{1}{2}z$.

Therefore $$\frac{(\cos r - \cos a)(\cos r - \cos b)}{(\cos r + \cos a)(\cos r + \cos b)} = \left(\frac{\cos r - \cos d}{\cos r + \cos d}\right)^2,$$

$$\frac{\cos^2 r - (\cos a + \cos b)\cos r + \cos a \cos b}{\cos^2 r + (\cos a + \cos b)\cos r + \cos a \cos b} = \frac{\cos^2 r - 2\cos d \cos r + \cos^2 d}{\cos^2 r + 2\cos d \cos r + \cos^2 d},$$

$$\frac{(\cos a + \cos b)}{\cos a \cos b + \cos^2 r} = \frac{2\cos d}{\cos^2 r + \cos^2 d}, \qquad \frac{\cos\beta \cos d}{\cos a \cos b + \cos^2 r} = \frac{\cos d}{\cos^2 r + \cos^2 d},$$

$$\frac{\cos\beta}{\cos a \cos b + \cos^2 r} = \frac{1}{\cos^2 r + \cos^2 d}.$$

Now eliminate $\cos a \cos b$ from this equation by the following method:—

$$\cos a = \sin\beta \sin d \cos\theta + \cos\beta \cos d,$$
$$\cos b = -\sin\beta \sin d \cos\theta + \cos\beta \cos d,$$
$$\cos a \cos b = \cos^2\beta \cos^2 d - \sin^2\beta \sin^2 d \cos^2\theta;$$

therefore $$\frac{\cos\beta}{\cos^2 r + \cos^2\beta \cos^2 d - \sin^2\beta \sin^2 d \cos^2\theta} = \frac{1}{\cos^2 r + \cos^2 d}.$$

By clearing of fractions and arranging, we have

$$\cos^2\beta(\cos^2 d + \cos^2\theta \sin^2 d) - \cos\beta(\cos^2 r + \cos^2 d) + (\cos^2 r - \cos^2\theta \sin^2 d),$$

$$\cos\beta = \frac{(\cos^2 r + \cos^2 d) \pm (\cos^2 r - \cos^2 d - 2\cos^2\theta \sin^2 d)}{2(\cos^2 d + \cos^2\theta \sin^2 d)},$$

$$\cos\beta = 1 \quad \text{or} \quad \frac{\cos^2 r - \cos^2\theta \sin^2 d}{\cos^2 d + \cos^2\theta \sin^2 d}.$$

If $\cos\beta = 1$, the points P and M coincide.

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If
$$\cos\beta = \frac{\cos^2 r - \cos^2\theta \sin^2 d}{\cos^2 d + \cos^2\theta \sin^2 d},$$
$$\frac{1-\cos\beta}{1+\cos\beta} = \frac{\cos^2 d - \cos^2 r + 2\cos^2\theta \sin^2 d}{\cos^2 r + \cos^2 d},$$
$$\tan^2 \tfrac{1}{2}\beta = \frac{\cos^2 d + \cos^2 r}{\cos^2 r + \cos^2 d} + \cos^2\theta \frac{2\sin^2 d}{\cos^2 r + \cos^2 d}.$$

Now, if we put $\sin^2\theta + \cos^2\theta$ for 1, and simplifying, we have
$$\tan^2 \tfrac{1}{2}\beta = \left(\frac{\sin^2 d - \cos^2 r}{\cos^2 r + \cos^2 d}\right)\cos^2\theta + \left(\frac{\cos^2 d - \cos^2 r}{\cos^2 d + \cos^2 r}\right)\sin^2\theta.$$

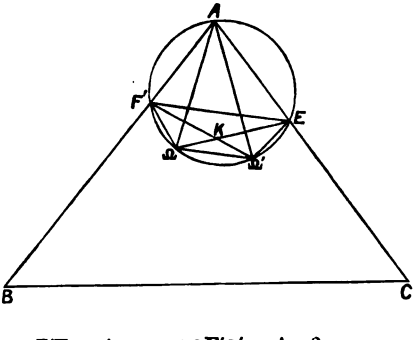
Therefore $A = \dfrac{\sin^2 d - \cos^2\gamma}{\cos^2\gamma + \cos^2 d}$ and $B = \cos^2 d + \cos^2 r$.

13817. (R. Tucker, M.A.)—Ω, Ω' K are the Brocard and symmedian points of a triangle. KΩ produced meets the sides BC, CA, AB in D, E, F, and KΩ' produced meets them in D′, E′, F′. Prove that the circles A$\Omega\Omega'$, B$\Omega\Omega'$, C$\Omega\Omega'$ respectively pass through E, F′; F, D′; D, E′; also that D′F, E′D, F′E are parallel to $\Omega\Omega'$. The radical axes of A$\Omega\Omega'$ (B$\Omega\Omega'$, C$\Omega\Omega'$) and of ABC are parallel. Prove also that
$$D'F : E'D : F'E = 1/\omega_b : 1/\omega_c : 1/\omega_a, \quad \text{where} \quad \omega_a \equiv a^4 - b^2c^2.$$

Solution by Rev. J. Cullen, B.A.; *and* Professor Sanjána, M.A.

Since
$\angle K\Omega\Omega' = \omega = \Omega'AE$,
the points Ω, Ω', E, A, are concyclic. In like manner also are Ω, Ω', F′, A. Hence
$\angle \Omega EF' = F'\Omega'\Omega = E\Omega\Omega$
therefore F′E is parallel to $\Omega\Omega'$.

The centre of the circle through A$\Omega\Omega'$ clearly lies on the Brocard diameter; therefore the radical axes are perpendicular to this line.



Again, $\angle F'\Omega E = \pi - A$, $\angle \Omega F'E = A - \omega$, $\angle \Omega F'\Omega' = A - 2\omega$;
therefore $F'E/\sin A = E\Omega/\sin(A-\omega) = \Omega\Omega'/\sin(A-2\omega)$;
therefore $F'E\omega_a = \Omega\Omega'\lambda^2$ $(\lambda^2 = \Sigma a^2 b^2)$;
therefore, &c.

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13634. (Professor NEUBERG.)—Connaissant les centres α, β, γ, δ des cercles circonscrits aux triangles BCD, ACD, ABD, ABC formés par quatre points A, B, C, D, trouver ces points A, B, C, D.

Solutions (1) *by* R. F. DAVIS, M.A.; (2) *by* Professor A. DROZ-FARNY, *and many others.*

(1) Upon reference to a figure it will be easily seen that A is the "isogonal conjugate" to α with respect to the triangle $\beta\gamma\delta$. Also B is the image of A in $\gamma\delta$, C of B in $\delta\alpha$, D of C in $\alpha\beta$. Hence A, B, C, D are determined.

(2) Soit A′ un point quelconque; son symétrique par rapport à l'axe $\gamma\delta$ sera B′. Le symétrique de B′ par rapport à l'axe $\alpha\delta$ sera C′. Celui de C′ par rapport à l'axe $\alpha\beta$ sera D′ et enfin le symétrique de D′ par rapport à $\beta\gamma$ sera A″. Si A′ se déplace dans le plan, les points A′ et A″ décrivent deux figures directement égales dont le point double est le sommet A cherché.

Si dans les figures A′ et A″ les côtés homologues ne sont pas parallèles, il y a une solution unique du problème. Il en est de même si ces côtés sont parallèles et de sens contraires. Si par contre les côtés homologues sont parallèles et de même sens, le problème est impossible.

13833. (W. E. JEFFARES, B.A.)—Show how to solve a quartic when the roots have the relation $\alpha^2\beta^2 = \gamma^2\delta^2$, and find the relation between the coefficients.

Solution by R. TUCKER, M.A.; *and* H. J. WOODALL, A.R.C.S.

Let the quartic be $x^4 - px^3 + qx^2 - rx + s = 0$.

Assume $\alpha + \beta = \lambda$, $\gamma + \delta = \mu$, $\alpha\beta = \pm\gamma\delta = \nu^2$ .....(i., ii., iii.);

then $\lambda + \mu = p$, $\lambda\mu = q - 2\nu^2$, $\lambda\mu = q$ .....(iv., v., v.a),

$s = \pm\nu^4$, $p\nu^2 = r$, $\nu^2(\mu - \lambda) = r$ ... (vi., vii., vii.a).

Since ν^2 is known from (vi.) or (vii.), therefore λ and μ are easily found, and thence the roots can be found. From (vi.), (vii.), $\pm sp^2 = r^2$.

13701. (Lt. Col. ALLAN CUNNINGHAM, R.E.)—Factorize

$N_1 = 64{,}000{,}000{,}000{,}729$ and $N_2 = 729{,}000{,}000{,}000{,}064$.

Solution by the PROPOSER, *and* C. E. BICKMORE, M.A.

Every number N of form $(x^6 + y^6)$ contains the factor $(x^2 + y^2)$, and the identity

$$\frac{x^6 + y^6}{x^2 + y^2} = (x^2 + 3xy + y^2)^2 - 6xy\,(x + y)^2$$

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gives the quotient $(x^6+y^6) \div (x^2+y^2)$ resolved into the *two* factors

$$(x^2+3xy+y^2) \mp \sqrt{(6xy)}\,(x+y),$$

which are both integers when $6xy = \square$.

Now $N_1 = 2^6 . 10^{12} + 3^6$; here $x = 2 . 10^2$, $y = 3$; $\sqrt{(6xy)} = 60$, and the quotient $(200^6+3^6) \div (200^2+3^2)$ contains the factors

$$(200^2+3 . 3 . 200+3^2) \mp 60\,(200+3) = 41,809 \mp 12,180;$$

$\therefore$ $N_1 = 40,009 . 29,629 . 53,989 = 40,009 . 29,629 . (13 . 4153)$.

Again, $N_2 = 3^6 . 10^{12} + 2^6$; here $x = 3 . 10^2$, $y = 2$; $\sqrt{(6xy)} = 60$, and the quotient $(300^6+2^6) \div (300^2+2^2)$ contains the factors

$$(300^2+3 . 2 . 300+2^2) \mp 60\,(300+2) = 91,804 \mp 18120;$$

$\therefore$ $N_2 = 90,004 . 73,684 . 109,924 = (4 . 22,501)(4 . 13 . 13 . 109)(4 . 27,481)$.

Note.—The identity used above is ascribed to Mr. Aurifeuille in Mr. Ed. Lucas's paper *Sur la Série récurrente de Fermat*, Rome, 1879, pp. 5, 6.

[Mr. Bickmore refers to the *Messenger of Mathematics*, Vol. xxvi., p. 8, for the quadratic partitions of the factors of $a^4-a^2b^2+b^4$, when $ab = 6R^2$, and points out that when this condition holds, and ab is prime to 13, a^6+b^6 is the product of those factors, one of which is a multiple of 13.]

13687. (Lt.-Col. Allan Cunningham, R.E.)—Factorize 1,000,000,000,000,000,027 and 27,000,000,000,000,000,001.

Solution by C. E. Bickmore, M.A.; *and the* Proposer.

The two numbers to be resolved are $10^{18}+3^3$ and $3^3 . 10^{18}+1$, and are consequently particular cases of $x^6+3^3y^6$, which is equal to

$$(x^2-3xy+3y^2)(x^2+3y^2)(x^2+3xy+3y^2).$$

When (as in these cases) x and y are prime to 7, $x^6 \equiv y^6 \equiv 1 \pmod 7$, by Fermat's theorem, and $1+3^3 = 28 \equiv 0 \pmod 7$; hence one of the three factors is a multiple of 7. In fact

$$10^{18}+3^3 = 997003 \times 1000003 \times 1003003 = 7^2 . 20347 . 1000003 . 1003003,$$

and
$$1+3^3 . 10^{18} = 2997001 \times 3000001 \times 3003001$$
$$= 7 . 428143 \times 853 . 3517 \times 31 . 73 . 1327,$$

as is found, on reference to factor tables for the first, third, and fourth millions.

[The Proposer observes "The identity $(a^3+b^3) \div (a+b) = (a+b)^2-3ab$, on which the resolution depends, along with many similar ones, is ascribed to Mr. Aurifeuille by Mr. Ed. Lucas, in his paper *Sur la Série récurrente de Fermat*, Rome, 1879. A list of twenty such formulæ is given on p. 6 of that memoir."]

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12399. (J. BRILL, M.A.)—Prove that, in the steady motion of a viscous incompressible fluid, if the vortex lines be so disposed that it is possible to cut them orthogonally by a series of surfaces, then the expression $p/\rho + V + \frac{1}{2}q^2$ is constant along a vortex line.

Solution by Professor S. SIRCOM, M.A.

Let $p/\rho + V + \frac{1}{2}q^2 = \chi$, ν the coefficient of viscosity. Then the equations of steady motion, in which $du/dt = dv/dt = dw/dt = 0$, are

$$-2v\zeta + 2w\eta = -d\chi/dx + \nu\nabla^2 u,$$

and two others.

Multiply ξ, η, ζ and add; then, since

$$\frac{dx}{ds}\Big/\xi = \frac{dy}{ds}\Big/\eta = \frac{dz}{ds}\Big/\zeta = \frac{1}{\lambda},$$

$$0 = -\lambda d\chi/ds + \nu(\xi\nabla^2 u + \eta\nabla^2 v + \zeta\Delta^2 w).$$

From $$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$$

we have $$\frac{d^2u}{dx^2} = -\frac{d^2v}{dx\,dy} - \frac{d^2w}{dx\,dz};$$

then $\nabla^2 u = d^2u/dz^2 - d^2w/dx\,dz - d^2v/dx\,dy + d^2u/dy^2 = 2(d\eta/dz - d\zeta/dy)$,

and the factor of 2ν becomes

$$\xi(d\eta/dz - d\zeta/dy) + \eta(d\zeta/dx - d\xi/dz) + \zeta(d\xi/dy - d\eta/dx)$$

$$= \eta^2\frac{d}{dx}(\zeta/\eta) + \zeta^2\frac{d}{dy}(\xi/\zeta) + \xi^2\frac{d}{dz}(\eta/\xi) \quad \ldots\ldots (1).$$

If $$\frac{dx}{ds}\Big/\xi = \frac{dy}{ds}\Big/\eta = \frac{dz}{ds}\Big/\zeta \quad \ldots\ldots (2)$$

is normal to a surface ϕ, then

$$\frac{d\phi}{dx}\Big/\xi = \frac{d\phi}{dy}\Big/\eta = \frac{d\phi}{dz}\Big/\zeta \quad \ldots\ldots (3),$$

and if all the vortex lines (2) are normal to ϕ, we obtain the required condition by eliminating ϕ by differentiation. Thus

$$\xi^2\frac{d}{dz}(\eta/\xi) = \xi^2\left(\frac{d\phi}{dx}\frac{d^2\phi}{dy\,dz} - \frac{d\phi}{dy}\frac{d^2\phi}{dx\,dz}\right)\Big/\left(\frac{d\phi}{dx}\right)^2,$$

and two others; then, adding and applying (3), we find (1) = 0 as the condition that all the vortex lines may be normal to a surface; hence in that case $d\chi'/ds = 0$, or χ' is constant along a vortex line.

13789. (W. H. SALMON, B.A.)—(1) Three circles touch a cardioid and pass through its pole. Show that their other three points of intersection are collinear. (2) If two circles touch a cardioid and pass through the pole, their common chord will bisect the angle between the two lines

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joining the pole to the points of contact of the circles with the cardioid. (3) The locus of the other extremity of the diameter through the pole of any circle touching a cardioid and passing through the pole is a circle.

Solution by Professor A. Droz-Farny.

L'inverse d'une parabole, le foyer étant pris comme centre d'inversion, est une cardioide ayant son pôle au foyer. Par conséquent les tangentes à la parabole se transforment suivant des cercles tangents à la courbe et passant par son pôle. La propriété (2) est une transformation directe de la propriété connue : Si d'un point, on mène des tangentes à une parabole, la droite qui joint ce point au foyer est la bissectrice de l'angle des rayons vecteurs aux points de contact.

Soient t une tangente à la parabole, coupant la tangente au sommet Δ en A, S le sommet de la parabole et F son foyer. On sait que FA est perpendiculaire sur t et que si l'on construit le point inverse A′ de A de manière que $FA \cdot FA' = FS \cdot FS' = r^2$ (r étant le rayon d'inversion) la circonférence décrite sur FA′ comme diamètre sera la transformée inverse de la tangente t. Or A′ appartient aussi à l'inverse de la tangente au sommet, la circonférence décrite sur FS′ comme diamètre : d'où la proposition (3). Enfin la propriété (1) est une conséquence de la propriété (3) et une application du théorème bien connu : Si d'un point F d'une circonférence, on mène trois cordes, FA′, FB′, FC′, les circonférences décrites sur ces cordes comme diamètres, se coupant deux à deux suivant trois points en ligne droite.

Remarque.—On démontrerait de la même manière la propriété suivante ; Les couples de cercles orthogonaux qui passent par le pôle d'une cardioide et sont tangents à la courbe ont leurs seconds points d'intersection sur une circonférence fixe, l'inverse de la directrice. On peut aussi démontrer la propriété (1) comme transformée de la propriété que la circonférence circonscrite a tout triangle ayant ses côtes tangents à une parabole passe par le foyer de la parabole.

[The Rev. J. Cullen solves the Question as follows :—

Since the inverse of a parabola is a cardioid, we get the results in the question by inverting the three following theorems, the focus being the centre of inversion. In a parabola (1) the circle passing through the points of intersection of three tangents passes also through the focus ; (2) the joins of the focus and the points of contact of two tangents are equally inclined to the join of their point of intersection and the focus ; (3) the locus of the point of intersection of a perpendicular from the focus on any tangent is the tangent at the vertex.]

Note on Question 13735.

In Col. Cunningham's solution the restriction to *positive* integers was overlooked. In each of his solutions, in both Cases (1) and (2), some one (or more) of α, β, α', β' is negative.

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13346 & 13363. (Professor P. H. Schoute.)—The ratios of the distances of a point in the plane of a triangle to the vertices A, B, C being given, find *two* points (P_1, P_2), and examine the correspondence between these points.

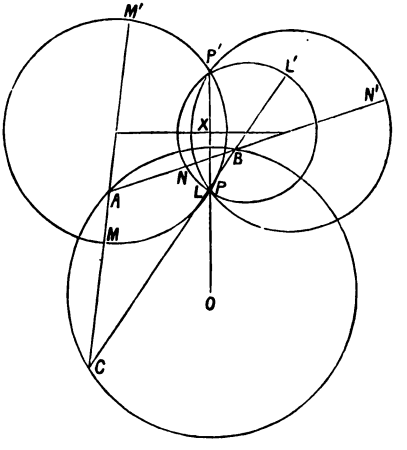
(13363.) (R. F. Davis, M.A.)—Prove the following construction for obtaining the pair of points (P, P′) satisfying the relation

$$PA : PB : PC = l : m : n \quad \text{(where } l, m, n \text{ are positive).}$$

Construct the straight line $l^2a\alpha + m^2b\beta + n^2c\gamma = 0$, and let fall upon it the perpendicular OX from the circumcentre. Upon OX take points P, P′, such that $OP + OP' = 2OX$, $OP \cdot OP' = OA^2$ (the criterion for real roots being that any two of the quantities la, mb, nc are together greater than the third).

Solution by W. E. Jeffares, B.A.; H. W. Curjel, M.A.; *and others.*

(13346.) Let the ratios be $l : m : n$. Divide BC, CA, AB in LL′, MM′, NN′, and internally and externally in the ratios $m : n$, $n : l$, $l : m$. The points P, P′ must clearly be at the common points of the three circles on LL′, MM′, NN′ as diameters, which are easily seen to be coaxal. Also, since the ranges (CB, LL′), (CA, MM′), (AB, NN′) are harmonic, the circumcircle of ABC cuts the three circles orthogonally; therefore P, P′ are inverse points with respect to the circumcircle ABC.



(13363.) That is, PP′ passes through the circumcentre O and

$$OP \cdot OP' = OA';$$

also OP cuts the line joining the middle points of LL′, MM′, NN′ (*i.e.*, $l^2a\alpha + m^2b\beta + n^2c\gamma = 0$) at right angles in X, such that

$$OP + OP' = 2OX.$$

Expressing the condition that the distance between the centres of two of the circles is less than the sum of their radii, we get $(al + mb)^2 > n^2c^2$; hence the condition for the reality of PP′ is that any two of la, mb, nc are together greater than the third.

[*Vide* solution of Quest. 10434, Vol. lxiv., p. 107.]

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398. (S. Watson.)—A semi-ellipse is placed in a vertical position, and a heavy particle is just set in motion at the highest point of the curve, and descends down it of its own weight; required the point at which it will leave the curve, and its velocity at that point.

Solution by H. W. Curjel, M.A.

Let the coordinates of the required point be $a\cos\phi$, $b\sin\phi$. Then

$$v^2 = 2ga(1-\cos\phi),$$

$$g\,\frac{b\cos\phi}{\sqrt{(b^2\cos^2\phi + a^2\sin^2\phi)}} = \frac{v^2}{\rho} = \frac{2ag(1-\cos\phi)}{[(b^2\cos^2\phi + a^2\sin^2\phi)^{\frac{3}{2}}]/ab};$$

therefore $$\cos\phi\,(b^2\cos^2\phi + a^2\sin^2\phi) = 2a^2(1-\cos\phi);$$

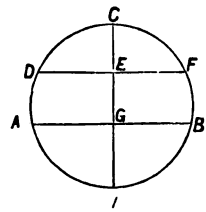
therefore $$(a^2-b^2)\cos^3\phi - 3a^2\cos\phi + 2a^2 = 0.$$

This determines ϕ and hence v.

12688. (D. Biddle.)—To divide a circle by parallel chords into three segments whose areas shall be in any required geometrical progression.

Solution by C. Bickerdike.

The following is an attempt to solve a particular case:—



Let area DEFC $= a$; let area DFAB $= 2a$; and area AGBI $= 4a$. Call radius r, and let CE $= x$. Then the area of

DEFC (a) $\quad \frac{4}{3}(2r)^{\frac{1}{2}}x^{\frac{3}{2}} - 2r^{-\frac{1}{2}}x^{\frac{5}{2}}$

$$- \frac{2r^{-\frac{1}{2}}x^{\frac{5}{2}}}{5\sqrt{2}} - \frac{2r^{-\frac{3}{2}}x^{\frac{7}{2}}}{28\sqrt{2}}, \text{ \&c.}$$

Now, by taking the first term of the series, which is near enough for our purpose, we have

$$\tfrac{4}{3}(2r)^{\frac{1}{2}}x^{\frac{3}{2}} = a; \text{ therefore } x = (9a^2/32r)^{\frac{1}{3}}.$$

Again, to find the versed sign GC $= x_1$, we have

$$\tfrac{4}{3}(2rx_1^3)^{\frac{1}{2}} = 3a, \quad \therefore \tfrac{16}{9}2rx_1^3 = 9a^2, \quad \therefore x_1 = 9^{\frac{1}{3}}(9r^2/32r)^{\frac{1}{3}};$$

$$\therefore \text{GC} - \text{EC} = \text{GE} = 9^{\frac{1}{3}}(9a^2/32r)^{\frac{1}{3}} - (9a^2/32r)^{\frac{1}{3}} = (9^{\frac{1}{3}}-1)(9a^2/32r)^{\frac{1}{3}},$$

and $$2r - \text{GC} = \text{GI} = 2r - 9^{\frac{1}{3}}(9a^2/32r)^{\frac{1}{3}};$$

$$\therefore \text{EC} = (9a^2/32r)^{\frac{1}{3}}, \quad \text{EG} = (9^{\frac{1}{3}}-1)(9a^2/32r)^{\frac{1}{3}}, \quad \text{GI} = \ldots 2r - 9^{\frac{1}{3}}(9a^2/32r)^{\frac{1}{3}}.$$

Note on Quest. 13557, Vol. lxviii., p. 75, *by* Rev. W. A. Whitworth, M.A.

To throw "all sixes in r throws with n dice" might mean any one of three things, but, whichever interpretation be adopted, the result can be written down at sight.

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(1) It might mean that nothing but sixes should be thrown.

$$\text{Chance} = (\tfrac{1}{6})^{rn}.$$

(2) It might mean that at some throw in the series n sixes should be thrown. The chance of failing at any throw $= 1-(\frac{1}{6})^n$;

$$\text{chance of failing altogether} = \left\{1-(\tfrac{1}{6})^n\right\}^r;$$

$$\text{chance of success} = 1-\left\{1-(\tfrac{1}{6})^n\right\}^r.$$

(3) It might mean that every particular face marked "six" should turn up in the course of the play.

$$\text{Chance for 1 die} = 1-(\tfrac{5}{6})^r; \qquad \text{chance for } n \text{ dice} = \left\{1-(\tfrac{5}{6})^r\right\}^n.$$

The last appears, from his solution, to be the interpretation which the Proposer intended.

13187. (Rev. J. Cullen, B.A.)—Find the form of ϕ so as to satisfy the functional equation $\phi(xyz) + \Sigma\phi(xy/z) = 4\Sigma\phi(x)$.

Solution by the Proposer.

Taking the x and y differentials and adding, we have

$$xyz^2\phi'(xyz) + xy\phi'(xy/z) = 2z\left\{x\phi'(x) + y\phi'(y)\right\} \quad \ldots\ldots\ldots\ldots(1).$$

Taking the same differentials of (1), we find

$$xyz^2\phi'(xyz) + x^2y^2z^3\phi''(xyz) + xy\phi'(xy/z) + x^2y^2/z\phi''(xy/z)$$
$$= 2z\left\{x^2\phi''(x) + x\phi'(x)\right\},$$
$$\ldots\ \ldots + \ldots\ \ldots + \ldots\ \ldots + \ldots\ \ldots$$
$$= 2z\left\{y^2\phi''(y) + y\phi'(y)\right\};$$

$$\therefore\ x^2\phi''(x) + x\phi'(x) = y^2\phi''(y) + y\phi'(y) = z^2\phi''(z) + z\phi'(z) = 2a^2,$$

where a is an arbitrary constant. The solution of

$$x^2\phi''(x) + x\phi'(x) = 2a^2 \quad \text{is} \quad \phi(x) = (a\log x)^2 + b\log x + c,$$

and, choosing $b = c = 0$, we have the required form of ϕ.

13752. (The late Professor Wolstenholme, M.A., Sc.D.)—In the two conics $U = 0$, $V = 0$, where

$$U \equiv kxy - z^2, \quad V \equiv (px-qy)(px+qy+z) + k'xy,$$

each equation has one variable parameter (k and k'); nevertheless their common chords all touch the same tricusp quartic $u^{-\frac{1}{3}} + v^{-\frac{1}{3}} + w^{-\frac{1}{3}} = 0$, where $u \equiv 2px + 2qy + \frac{5}{2}z$, $v \equiv px + qy - z + \sqrt{3}(px-qy)$,

and $w \equiv px + qy - z - \sqrt{3}(px-qy)$.

The same envelope would be got by variation of k alone, or of k' alone;

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or we might obtain it as the envelope of the tangent at the point of contact when U, V touch one another. Explain how it is possible to have a definite envelope when any common chord must depend on the two parameters k, k'.

[It will be found that the equation of any common chord may be written

$$\lambda^3 qy - \lambda^2(px+z) + \lambda(qy+z) - px = 0;$$

λ no doubt being a function of k and k'. The cusps occur when λ has the values 1 or $-2 \pm \sqrt{3}$; and, at any point on the envelope, u, v, w are in the ratios

$$(\lambda^2 - 4\lambda + 1)^2,\ (\lambda+1)^2[1+\lambda+\sqrt{3}(1-\lambda)]^2,\ (\lambda+1)^2[1+\lambda-\sqrt{3}(1-\lambda)]^2,$$

whence the equation as given is immediately deduced. An interesting particular case is when U is a conic with given focus and directrix, and V a circle touching the axis of U in the focus. The common chord of *any* pair touches a curve, which has been already discussed as an envelope in two different ways both distinct from this. *Educational Times*, Quest. 7220, December, 1883; see Vol. XL., p. 115.]

Solution by W. C. STANHAM, B.A.

The common chords of $U = 0$ and $V = 0$ are the common chords of $U = 0$ and $(px-qy)(px+qy+z) + (k'/k)z^2 = 0$ (or $W = 0$).

$U = 0$ and $W = 0$ are conics whose asymptotes are fixed and independent of k and k', and, since the mid-point of a chord of a conic is the mid-point of the intercept of the chord (produced both ways) cut off by the asymptotes of the conic, any common chord (produced) of $U = 0$ and $V = 0$ is a straight line the mid-point of whose intercept by $x = 0$ and $y = 0$ is the mid-point of its intercept by $px - qy = 0$ and $px + qy + z = 0$. Clearly all such straight lines have a definite envelope independent of k and k'.

(When $U = 0$ and $V = 0$ touch, the common tangent at the point of contact is in a limiting position of a common chord. It therefore has the same definite envelope.)

13714. (J. C. WASSON.)—A variable circle rolls round the circumference of a fixed circle, the contact being internal. If the linear dimensions of the variable circle are $\frac{2}{3}$ or $\frac{4}{3}$ those of the fixed circle, every diameter of the former considered rigidly attached to it has a motion identically similar to that of a SIMSON'S line.

Solution by Rev. J. CULLEN, B.A.; W. J. DOBBS, M.A.; *and the* PROPOSER.

The envelope of the diameter of a circle rolling on any fixed curve is the roulette traced by a point on the circumference of a circle of half the radius rolling on the same curve.

Now, if the fixed curve be a circle of radius a, and that of the rolling circle b, then the tangent to the hypocycloid is

$$x\cos\omega + y\sin\omega = (a-2b)\sin a\omega/(a-2b).$$

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Putting $b = a/3$ or $2a/3$, we get $x\cos\omega + y\sin\omega = a\sin 3\omega/3$; so that the envelope is a three-cusped hypocycloid, which is also the envelope of the SIMSON line if the fixed circle is concentric with the nine-point circle. Hence the motion of the diameter and the SIMSON line are identically similar (see Quest. 5600, Vol. XXIX., p. 82).

3975. (P. O'CAVANAGH.)—If a double point and also a point of inflexion of a cubic be projected to infinity, what will be the nature of the new projected cubic?

Solution by Prof. RAMACHANDRA ROW, M.A.

Since the double point is at infinity, the tangents therefrom become parallel asymptotes; taking this as the direction of the axis of y, the equation of the asymptotes is $x^2 + ax + b = 0$ (either real or imaginary). Let the axis of x pass through the point of inflexion and the tangent thereat.

The equation of the projected cubic must be of the form

$$y(x^2 + ax + b) = lx + my + n.$$

Again $y = 0$ must not cut the curve at points at finite distance; therefore $l = 0$ and $n = 0$; therefore the curve becomes $y(x^2 + ax + b) = my$, *i.e.*, the projected cubic consists of three straight lines one of which is real and the other two must be both real or both imaginary.

4086. (A. RENSHAW.)—If AC be the chord of a heptagon inscribed in a circle, diameter AB ($= 2r$), and AB be taken as the axis of x, and a line at right angles to it through A as that of y, then, if (x', y') be the coordinates of C, prove that

$$-\frac{2y'(2x'-r)\{x'(x'-r)-y'^2\}}{\{x'(x'-r)-y'^2\}+y'^2(2x'-r)^2} = \frac{x}{y}.$$

Solution by H. J. WOODALL, A.R.C.S.

Call CAB θ and a the angle of the heptagon ($= \frac{5}{7}\pi$); then θ will have one of the values $a/10$, $3a/10$, or $5a/10$. Hence

$$7\theta = \tfrac{1}{2}\pi,\ \tfrac{3}{2}\pi,\ \text{or}\ \tfrac{5}{2}\pi,\ \text{and}\ \tan 7\theta = \infty.$$

But $\tan 7\theta\{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta\} = \tan\theta(7 - \&c.)$.

Hence, if $\tan 7\theta = \infty$, but $\tan\theta \neq 0$, we must have

$$1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta = 0.$$

For $\tan\theta$ write y/x, and we get

$$x^6 - 21x^4y^2 + 35x^2y^4 - 7y^6 = 0,$$

to which, I am sorry to say, it appears impossible to reduce the equation given in the enunciation (bearing in mind that $x^2 + y^2 - 2xr = 0$).

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13677. (W. E. JEFFARES, B.A.)—If α, β, γ be the roots of

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

find the equation whose roots are

$$(\beta-\gamma)^2(2\alpha-\beta-\gamma)^2, \quad (\gamma-\alpha)^2(2\beta-\gamma-\alpha)^2, \quad (\alpha-\beta)^2(2\gamma-\alpha-\beta)^2.$$

(See BURNSIDE and PANTON'S *Theory of Equations*, pp. 89 and 115.)

Solution by Rev. J. CULLEN, B.A.; A. S. TOMBE; *and many others.*

The equation of squared differences is

$$x^3 + \frac{18H}{a^2}x^2 + \frac{81H^2}{a^4}x + \frac{27}{a^6}(G^2 + 4H^3) = 0.$$

Also, since $(\gamma-\alpha)^2-(\alpha-\beta)^2 = (\beta-\gamma)(2\alpha-\beta-\gamma)$, &c., the required equation is $a^{12}x^3 - 162a^8H^2x^2 + 6561a^4H^4x + 19683G^2(G^2 + 4H^3) = 0.$

8272. (EMILY PERRIN.)—Given a triangle ABC and any point P in its plane, give a geometrical construction to determine the line through P which shall bisect the area of the triangle.

Solution by NOMINIS UMBRA.

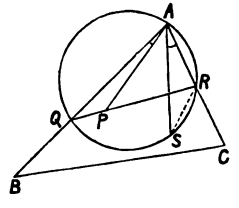
Analysis.—Let P be the given point, and let required line be PQR. At A make the angle RAS = PAQ, and meeting the circumcircle of QAR in S. Then, since rectangle

$$AP \cdot AS = AQ \cdot AR = \tfrac{1}{2} \cdot AB \cdot AC,$$

the point S is given. Also, the angle

$$PRS = QAS.$$

Hence the point R is known, its locus being a known segment of a circle on the line PS. The line PR gives the required solution.



Note.—A similar method, of course, solves the problem to cut off any required area from the given triangle.

[For another solution of this Question see Vol. XLIV., p. 92.]

13209. (Rev. Dr. BATTESON WRIGHT.)—To draw an angle of 10°, geometrically.

Solution by W. E. HEAL.

The problem is impossible; otherwise it would be possible to inscribe a regular nonagon in a given circle.

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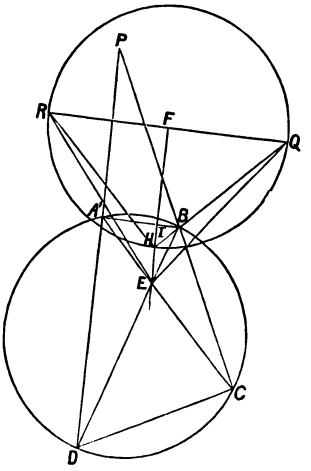
13384. (W. J. C. Miller, B.A.)—In a given circle inscribe a quadrilateral having one side of given length, the two adjacent sides passing through a given point, and the rectangle under the distances from two given points to the intersection of its diagonals a minimum.

Solution by the late Morgan Brierley.



Let ABCD be the given circle, P the given point through which the adjacent sides pass, and Q, R the other two given points.

Join Q, R, and parallel to the joining line draw AB in the given circle equal in length to the given side, and through its extremities draw the lines PAD, PBC; also join DC; ABCD will be the quadrilateral required.

For, let fall the perpendicular EF, from the intersection E of the diagonals AC, BD, cutting AB in I and a circle described on QR in H. Join Q, H; R, H; then

$$RH \cdot QH = QR \cdot FH;$$

and QE . ER = rectangle under EF and diameter of the circle passing through E, Q, R; and EF is obviously the shortest line that can be drawn from E to the line QR; therefore QE . ER is a minimum.

13746. (Lionel E. Reay, B.A.)—Find the area of the triangle with sides equal to the medians of a given triangle. Show whether such a triangle is always possible.

Solution by Prof. K. J. Sanjána, M.A.; W. J. Dobbs, M.A.; *and others.*

Let a, b, c be the sides, g, h, k the medians; then we can prove

$$16\,(g^2h^2 + h^2k^2 + k^2g^2) = 9\,(a^2b^2 + b^2c^2 + c^2a^2),$$

and

$$16\,(g^4 + h^4 + k^4) = 9\,(a^4 + b^4 + c^4).$$

Hence the area of triangle formed by the medians is $\sqrt{\frac{9}{16}}$ or $\frac{3}{4}$ of the given triangle. As $2\,(g^2h^2 + h^2k^2 + k^2g^2) - (g^4 + h^4 + k^4)$ is positive, if $2\,(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$ is so, the triangle is always possible if the original one is possible.

[The Proposer solves the Question as follows:—As in Quest. 13713

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(see fig.), $\alpha\beta = \frac{1}{3}AD$; hence the triangle $\alpha\beta\gamma = \frac{1}{9}$ of the triangle formed by the medians.

Also $A\alpha\gamma = B\beta\alpha = C\gamma\beta = \frac{1}{3}ABC$. Therefore $\alpha\beta\gamma = \frac{1}{3}ABC$; therefore the triangle formed by the medians $= \frac{2}{4} \times \frac{1}{3}ABC = \frac{3}{4}ABC$. Since the construction is always possible, the triangle is also possible.

13501. (W. J. C. Miller, B.A.)—Through a point, taken at random within a circle, a chord is drawn at random to cut two fixed diameters of the circle; find the probability of the triangle thus formed being acute-angled.

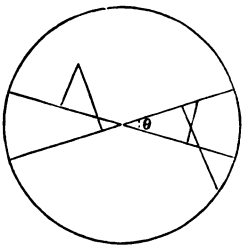
Solution by L. E. Reay, B.A.

Let the fixed diameters contain an acute angle θ.

Take any point within the circle, and draw perpendiculars from it on to the diameters. These also contain an acute angle θ.

Any chord through the point which makes an acute-angled triangle with the diameters must fall within the acute angle formed by these perpendiculars.

The chance for this is θ/π.



4653. (Matthew Collins, LL.B.)—Let AC be the side of a regular pentagon inscribed in a circle whose centre is O; on AO draw an equilateral triangle AOB; bisect the arc BC in D, and the arc AD in E; then prove that the chord AE will be very nearly equal to the side of a regular undecagon inscribed in the circle.

Solution by K. S. Putnam.

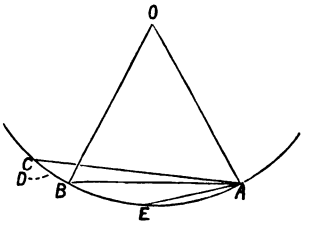
Let AC be the side of the pentagon inscribed in the circle whose centre is O (not D), AOB the equilateral triangle, D the mid-point of the arc BC, and E the mid-point of the arc AD.

Arc AC $= 72°$; AB $= 60°$;

BC $= 72° - 60° = 12°$; BD $= 6°$;

AD $= 60° + 6° = 66°$; arc AE $= 33°$.

But the side of the undecagon subtends $360° \div 11 = 32\frac{8}{11}°$. The chord AE is therefore a trifle too long for the side of the regular inscribed undecagon.



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